Gridspace

IAP Program 2023
Digital Signal Processing (DSP)

January 11th 2023

2. 1. 3.

D.S.P.

Digital Signal Processing

Signal

1.
INFORMATION OVER A CHANNEL

"In **signal processing**, a signal is a function that conveys information about a phenomenon."

Roland Priemer

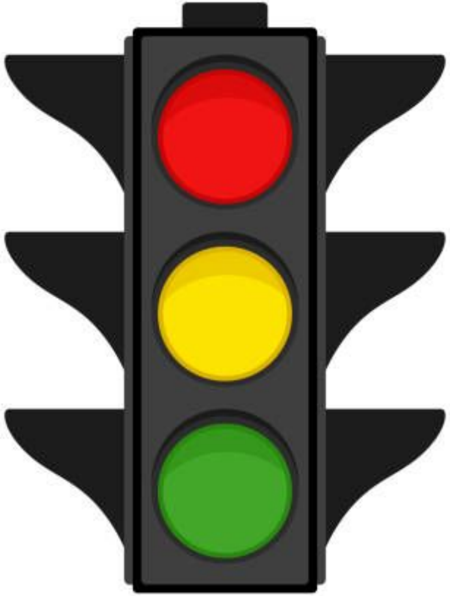
Definition of a signal

$$f(t) = \int_0^\infty \left(a(\lambda)\cos(2\pi\lambda t) + b(\lambda)\sin(2\pi\lambda t)\right) d\lambda.$$

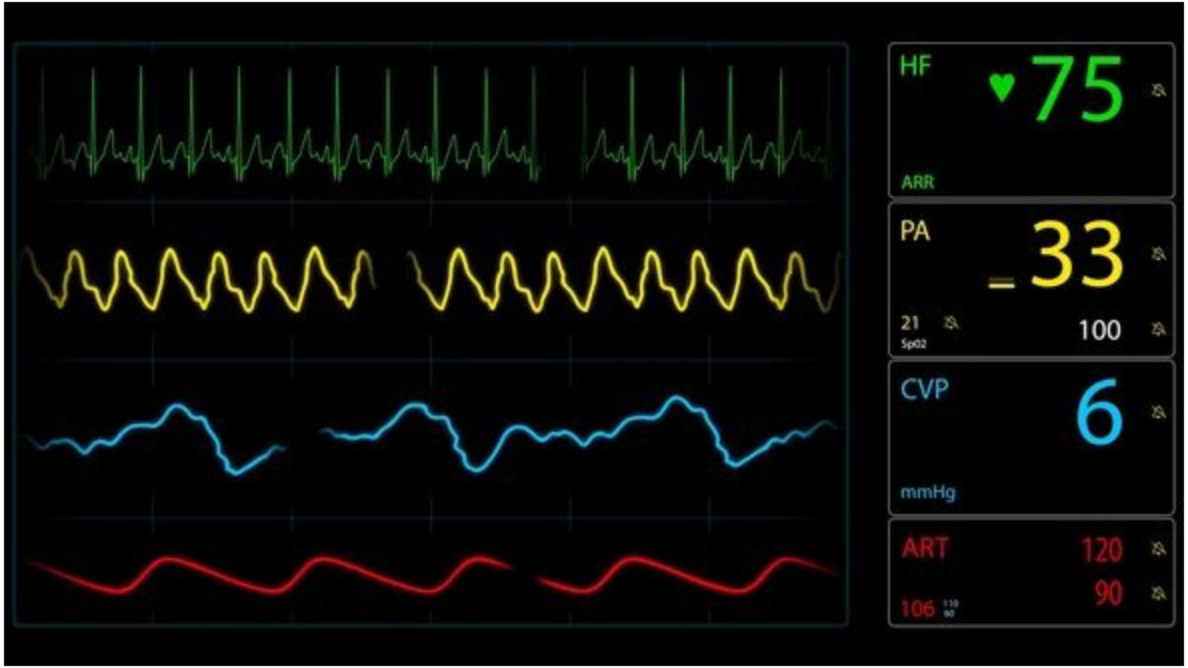
A signal

Simple Signals

Simple Signals

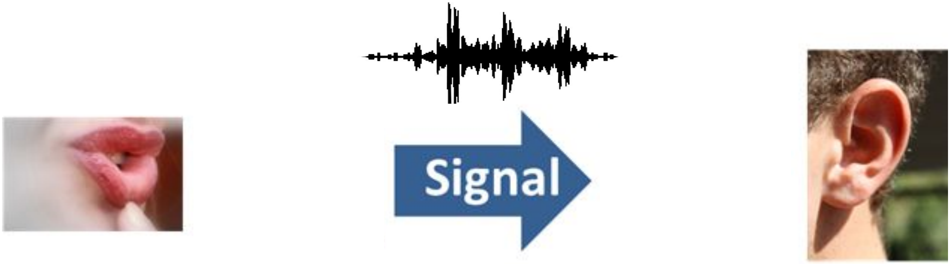


Simple Signals

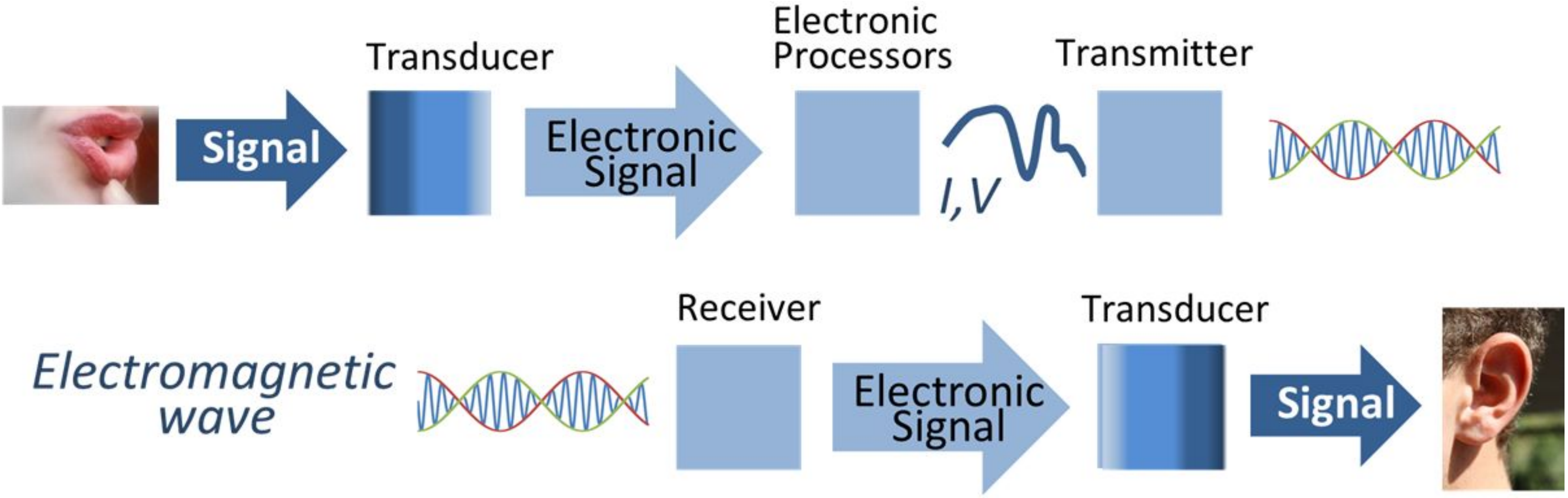


ECG: 1D time-varying electrical signal

Image: 2D spatial signal



The people in the room where it happens

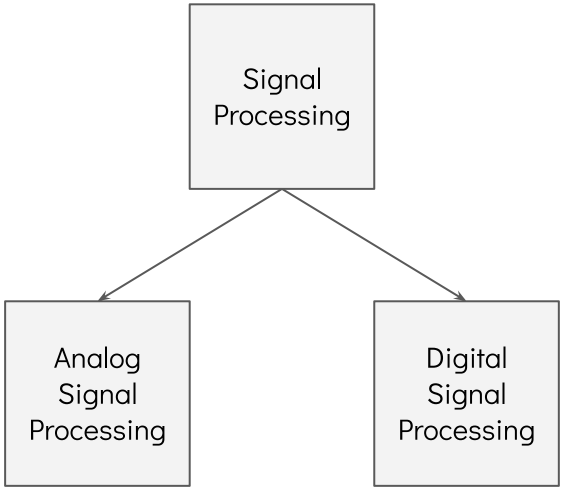


The people not in the room where it happens

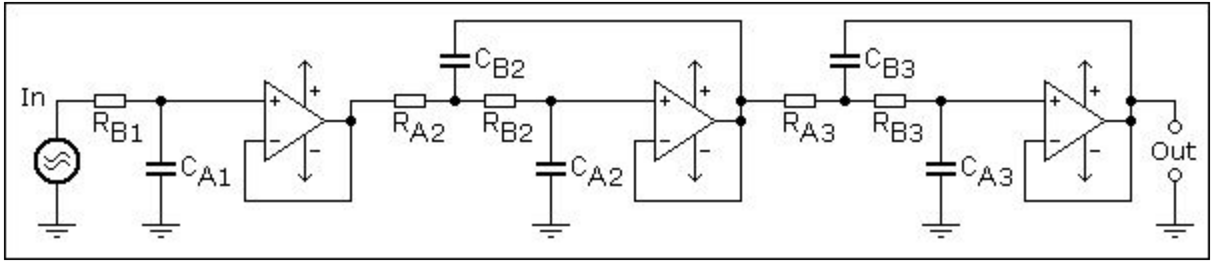
"In general, the term signal processing refers to the science of analyzing time-varying physical processes."

Richard Lyons

Definition of signal processing



Analog and Digital Signal Processing



Analog Signal Processing

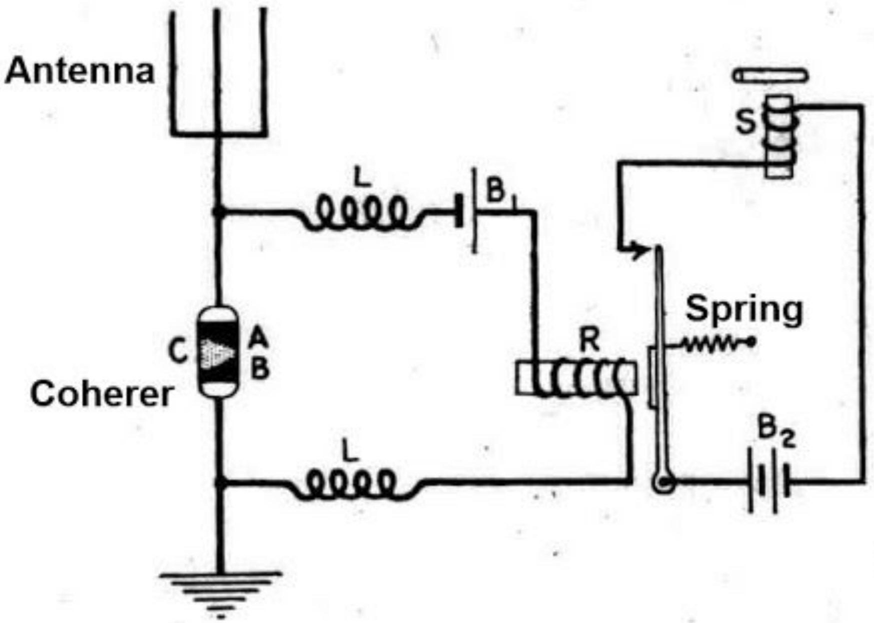
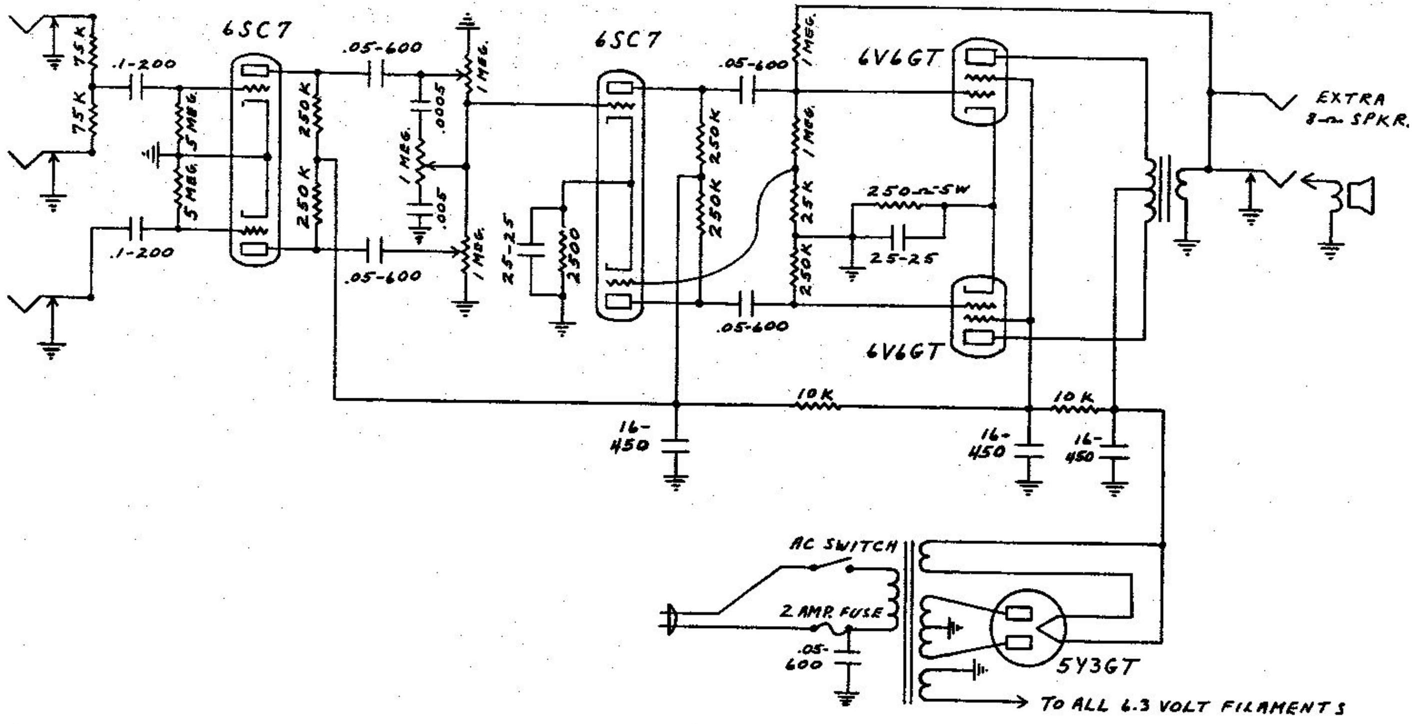


Fig. 101. Marconi 1896 Receiver.

Early Radio and ASP

FENDER "DELUXE" SCHEMATIC

MODEL 5C3



Example: Guitar Amps

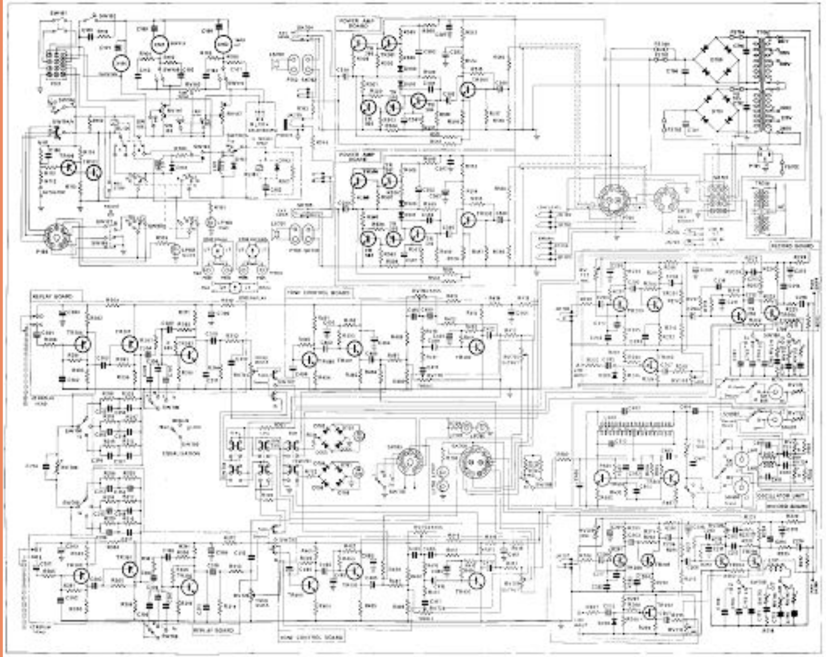
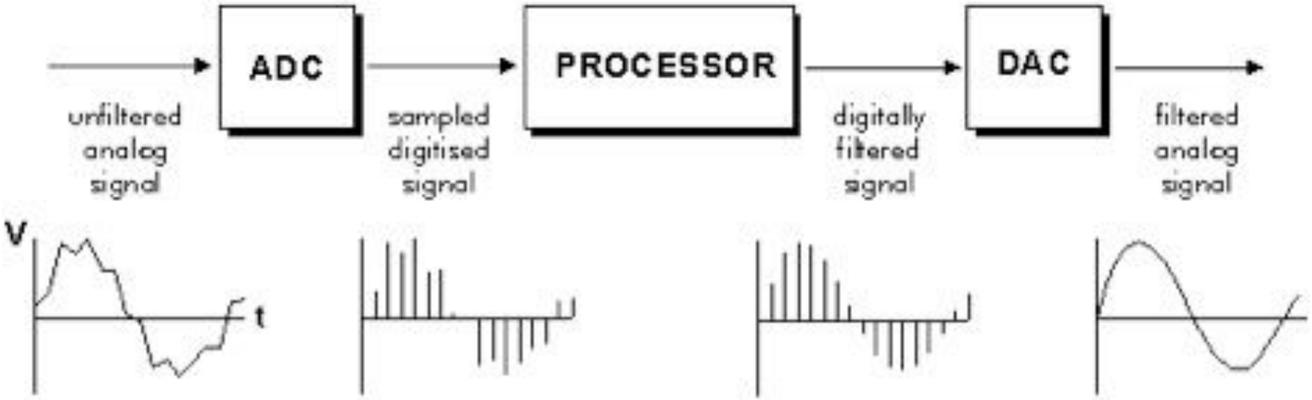


FIG 29 CIRCUIT DIAGRAM OF RECORDER 250-060 ISSUE 2

VS

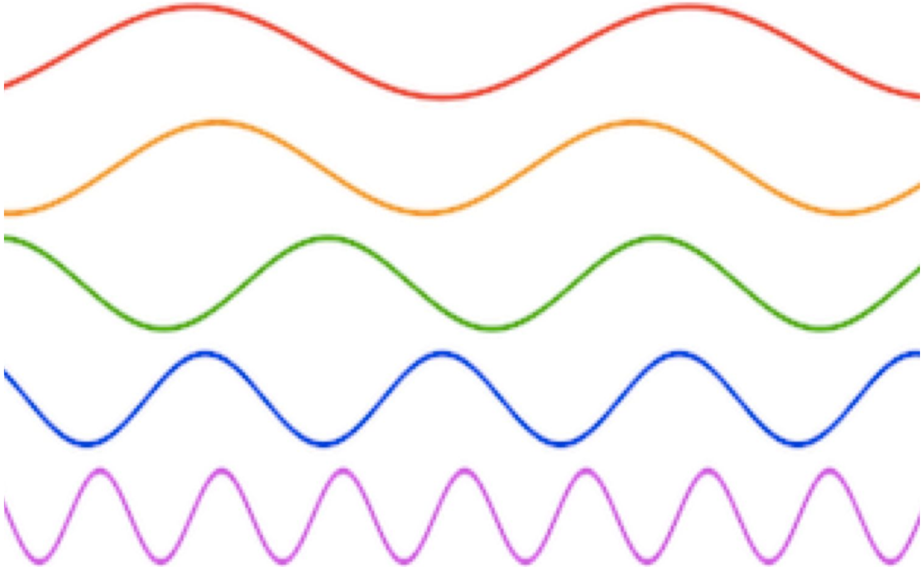
```
for (int i=0; i<num_samples; i++) {
        for (int j=0; j<filter_length, j++) {
                samples[i+j]*filter[j];
        }
}
```

The Signal as Data



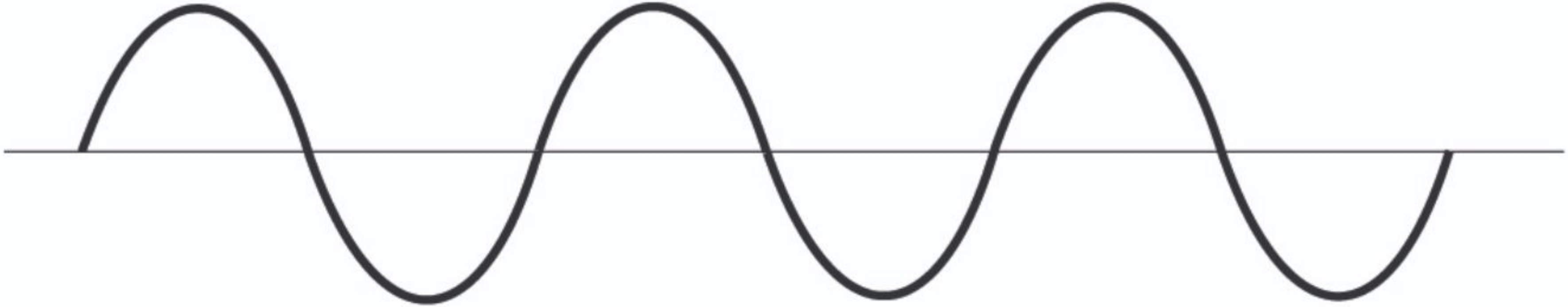
Digital Signal Processing

2.
THE SINE WAVE



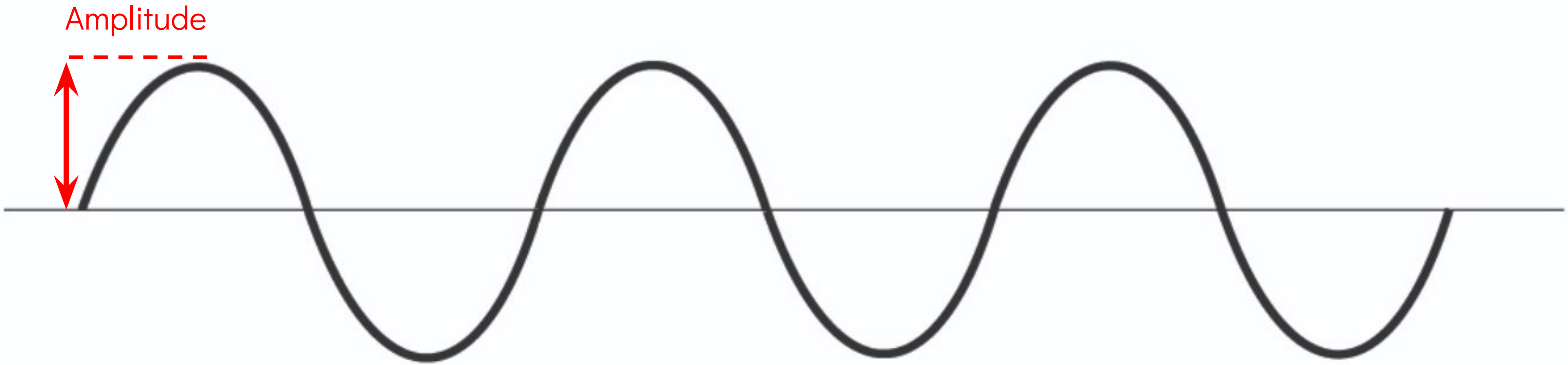
The Sine Wave as a Signal Primitive

$$x(t) = A\sin(2\pi f t + \varphi)$$



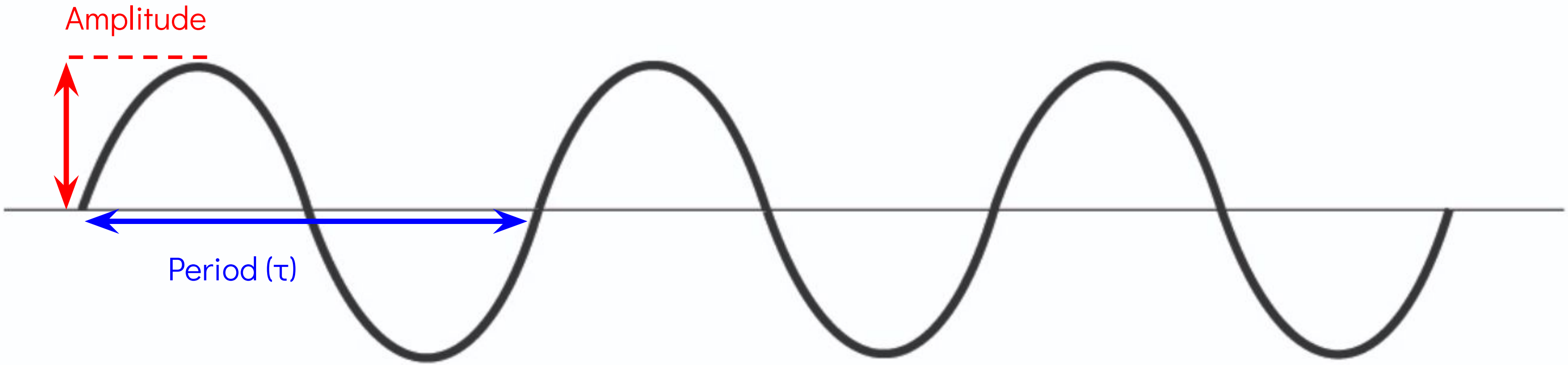
A Single Pure Frequency No Harmonics

$$x(t) = \underline{A}\sin(2\pi f t + \varphi)$$



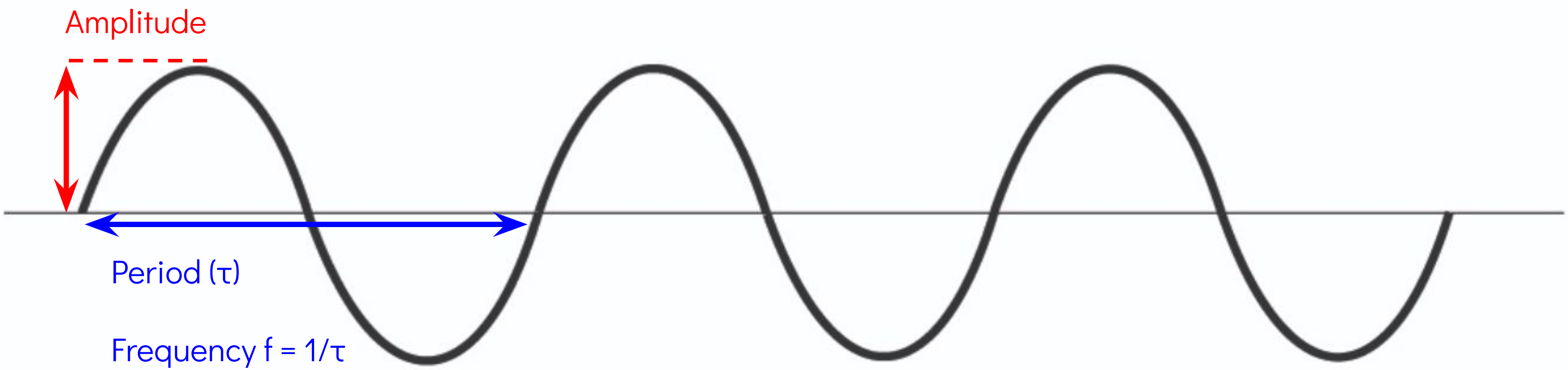
Fundamental Signal Vocabulary

$$x(t) = \underline{A}\sin(2\pi f t + \varphi)$$



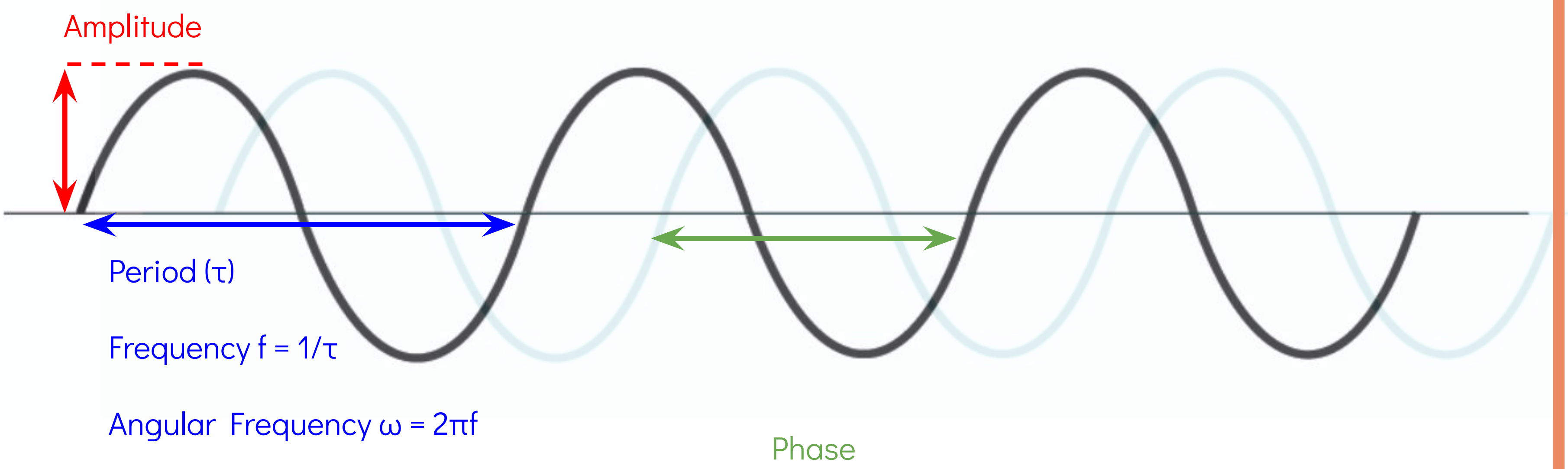
Fundamental Signal Vocabulary

$$x(t) = \underline{A}\sin(2\pi \underline{f} t + \varphi)$$



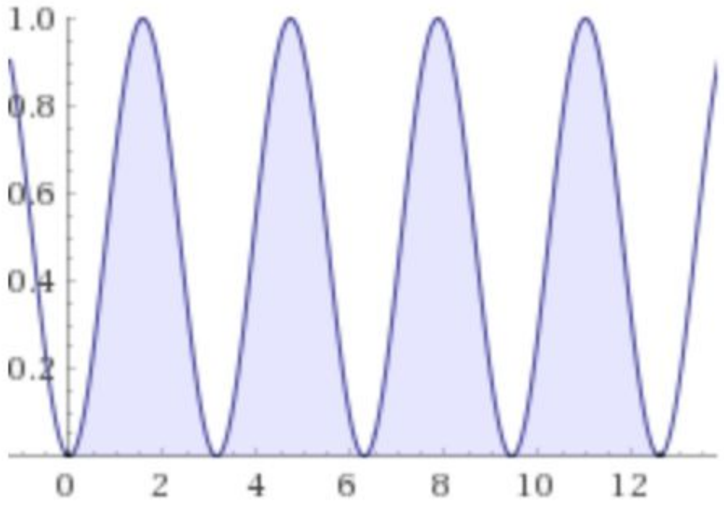
Fundamental Signal Vocabulary

$$x(t) = A\sin(2\pi f t + \varphi)$$



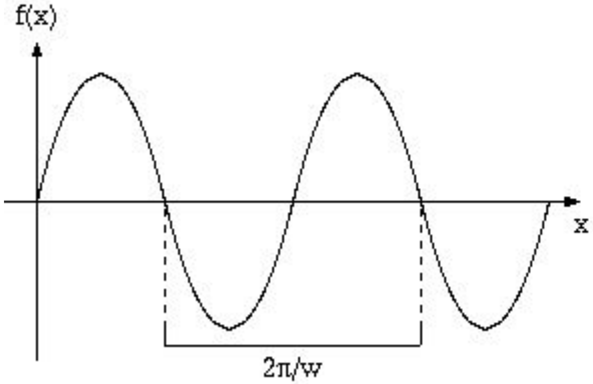
Fundamental Signal Vocabulary

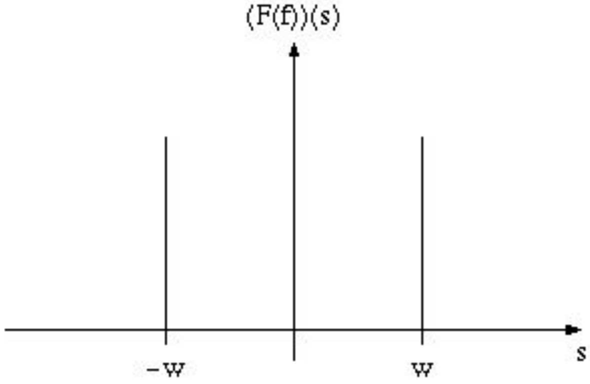
$$E = A \int \sin^2(2\pi f t + \varphi)$$



Energy

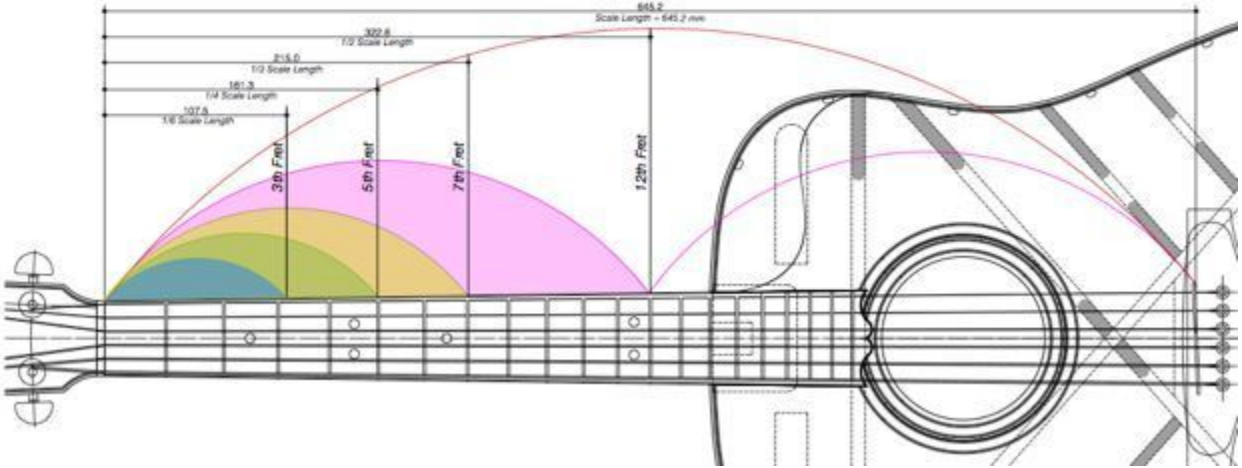
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(x)\,e^{i\omega x}\,dx = i\sqrt{\frac{\pi}{2}}\,\delta(\omega-1) - i\sqrt{\frac{\pi}{2}}\,\delta(\omega+1)$$



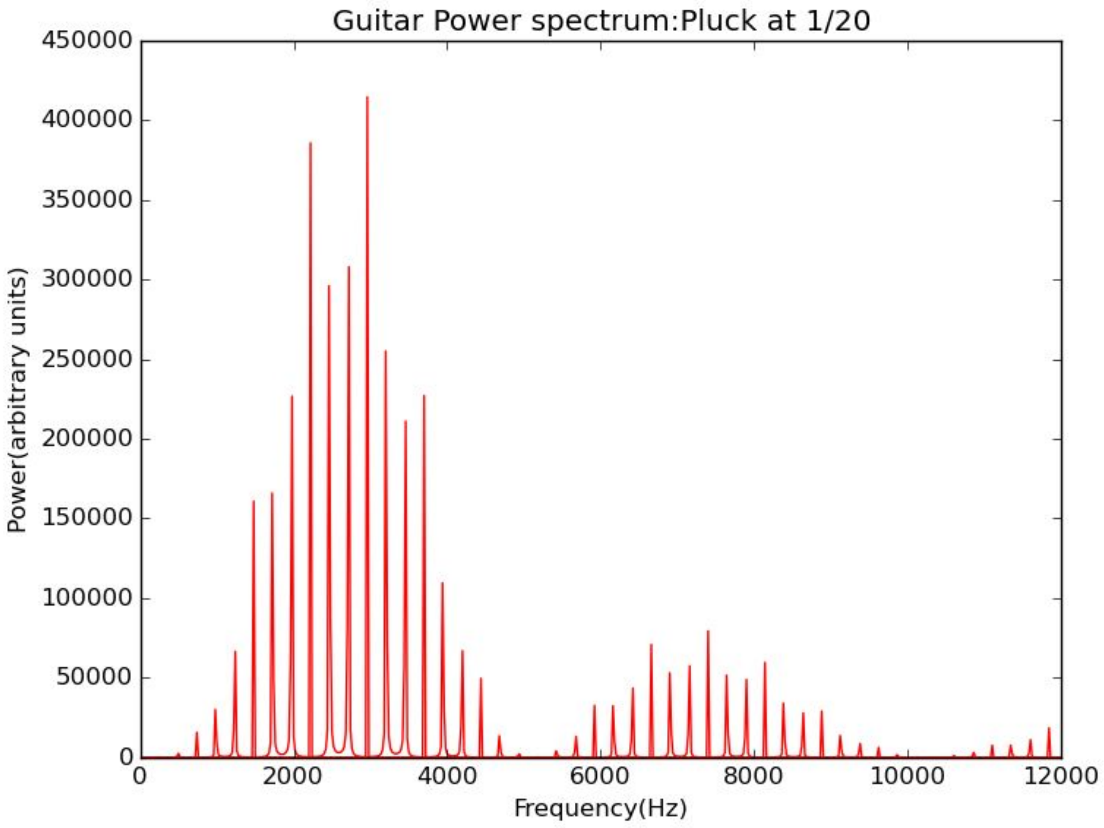


Frequency Domain

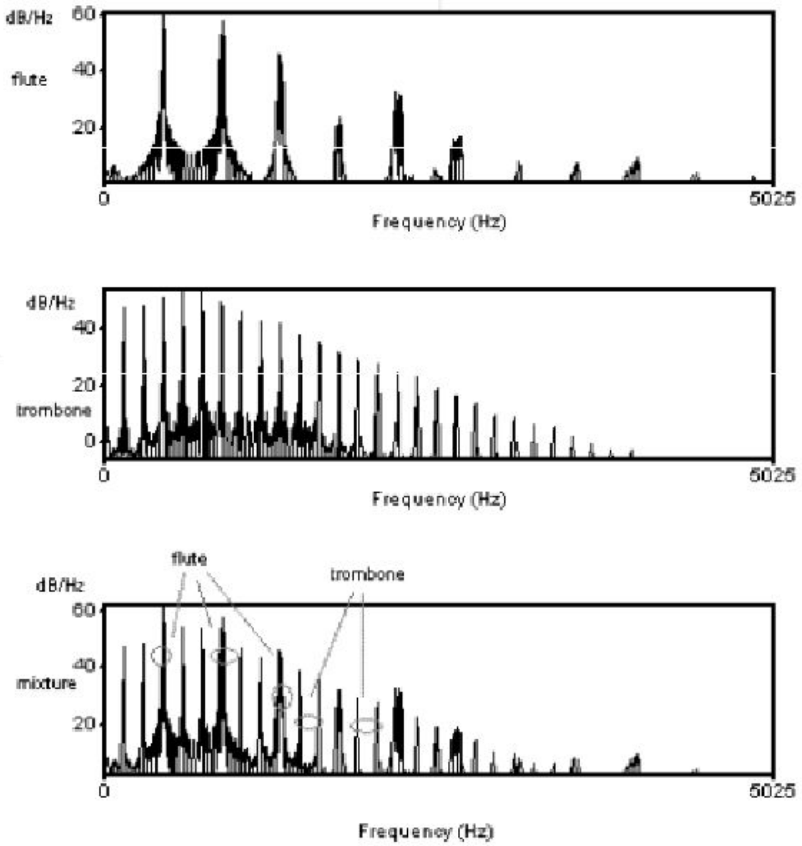
3.
REAL WAVES



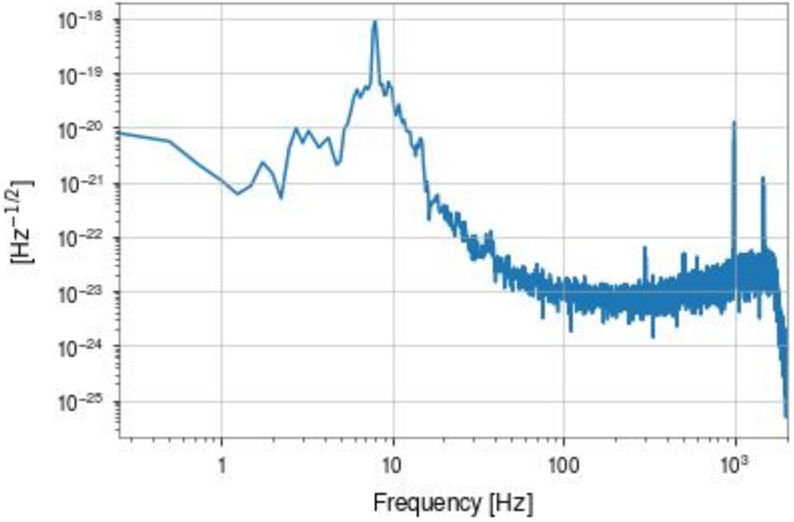
Example: Generating Real Waves



Frequency Domain of a Vibrating String



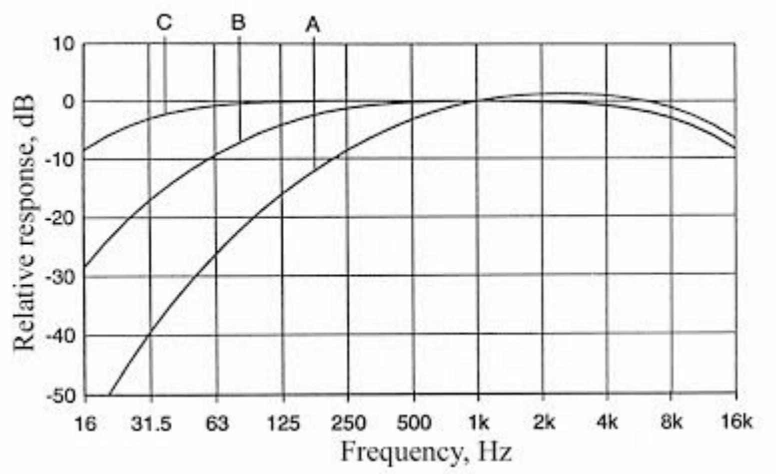
Distinct Harmonic Composition and Complex Spectra



Log Power Spectrum

$$10log(P/P_0)$$

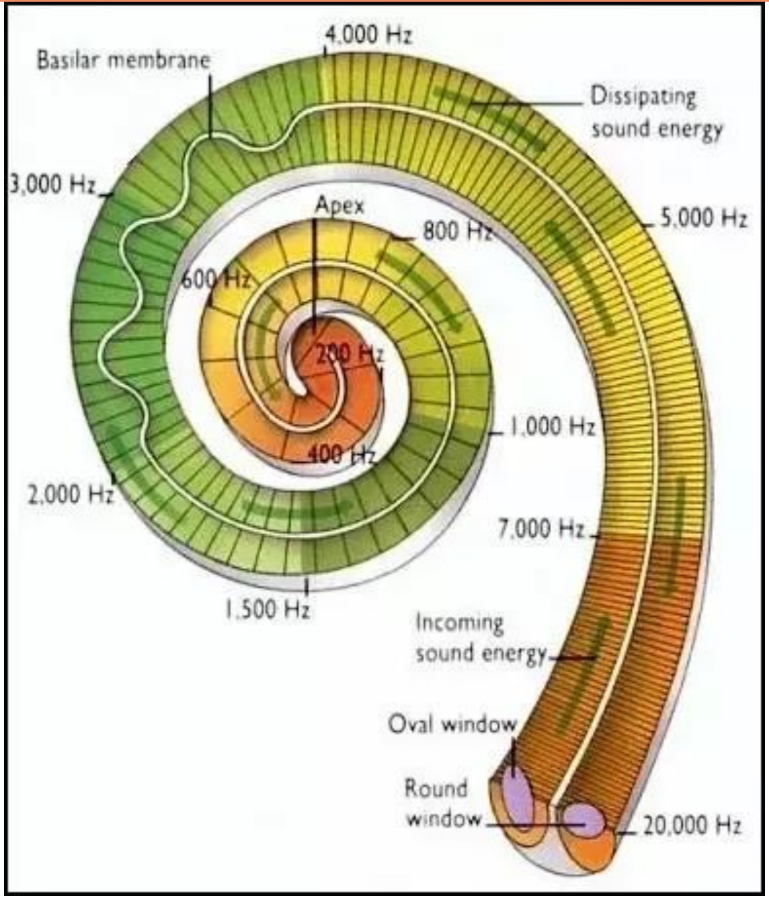
The Decibel (dB)



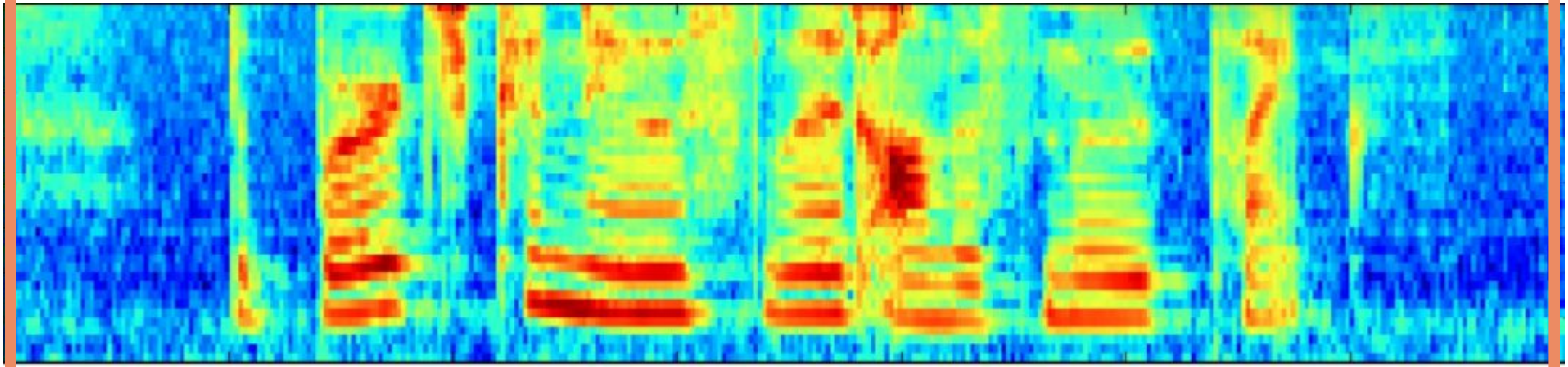
The Importance of a Reference dBa, dBb, dBc

$$m = 2595 \log_{10}\left(1 + \frac{f}{700}\right)$$

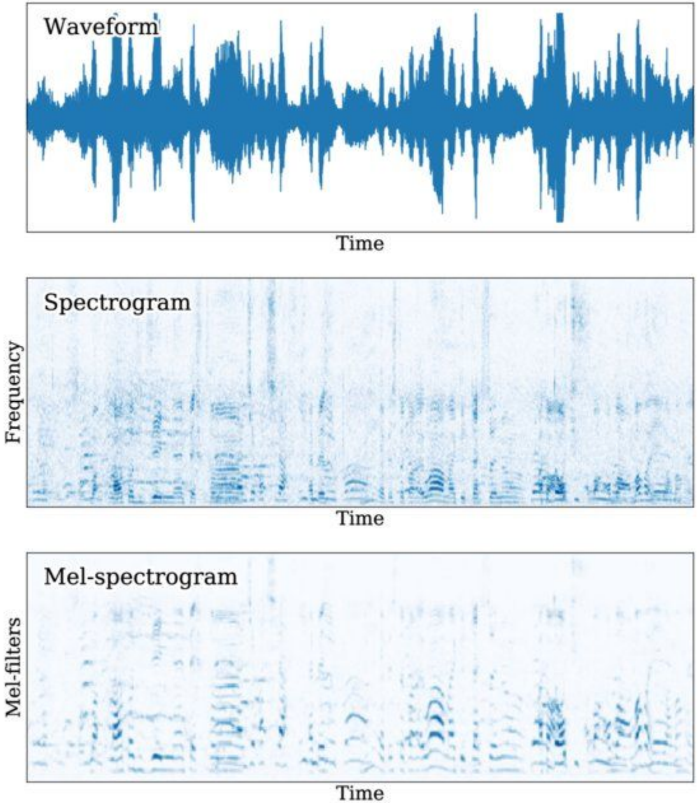
The Mel


4,000 Hz
Basilar membrane
Dissipating
sound energy
3,000 Hz
Apex
800 Hz
5,000 Hz
600 Hz
200 Hz
1,000 Hz
400 Hz
2,000 Hz
7,000 Hz
1,500 Hz
Incoming
sound energy
Oval window
Round
window
20,000 Hz

The Cochlea

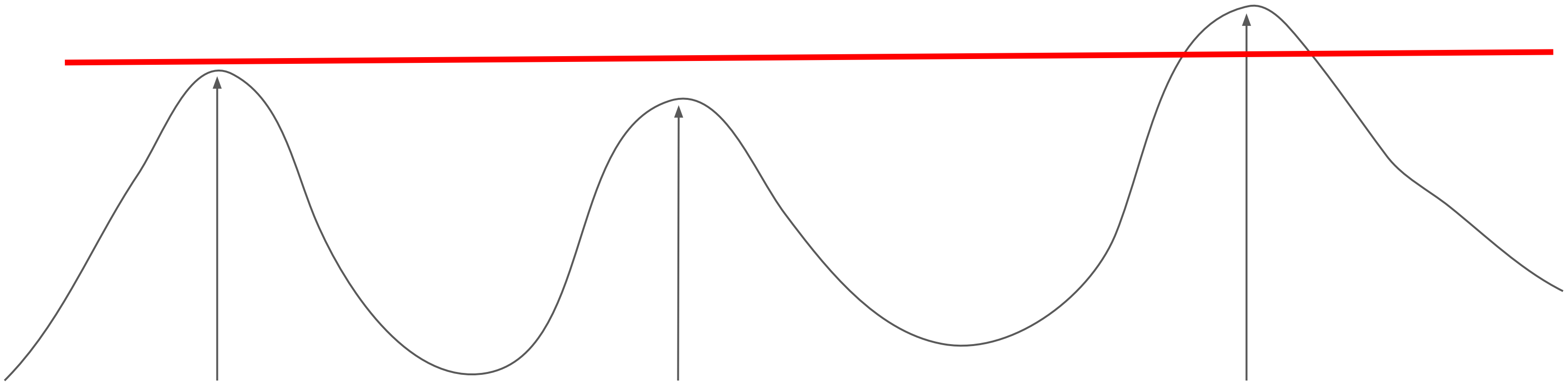


The Mel Spectrogram

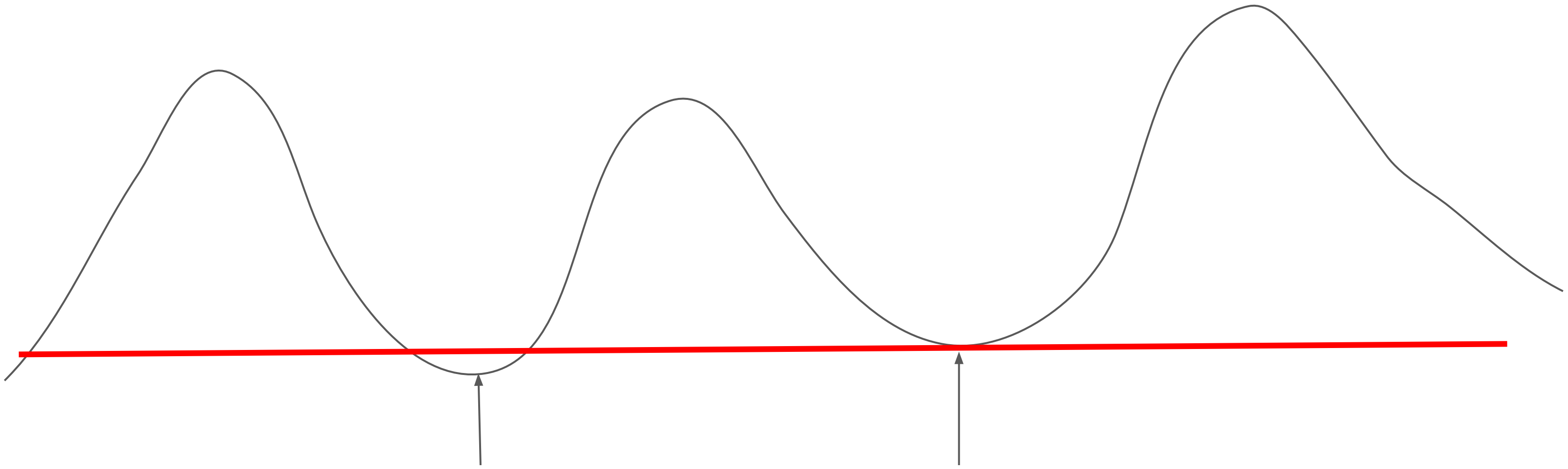

Waveform
Time
Spectrogram
Frequency
Time
Mel-spectrogram
Mel-filters
Time

4.
SAMPLING

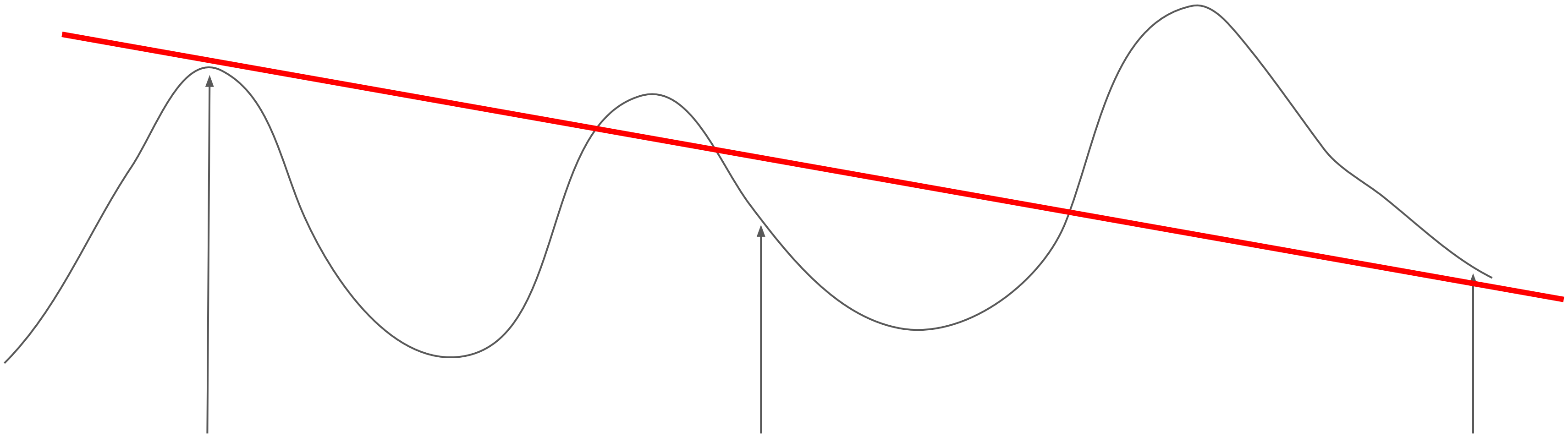
Example: A Weather Station



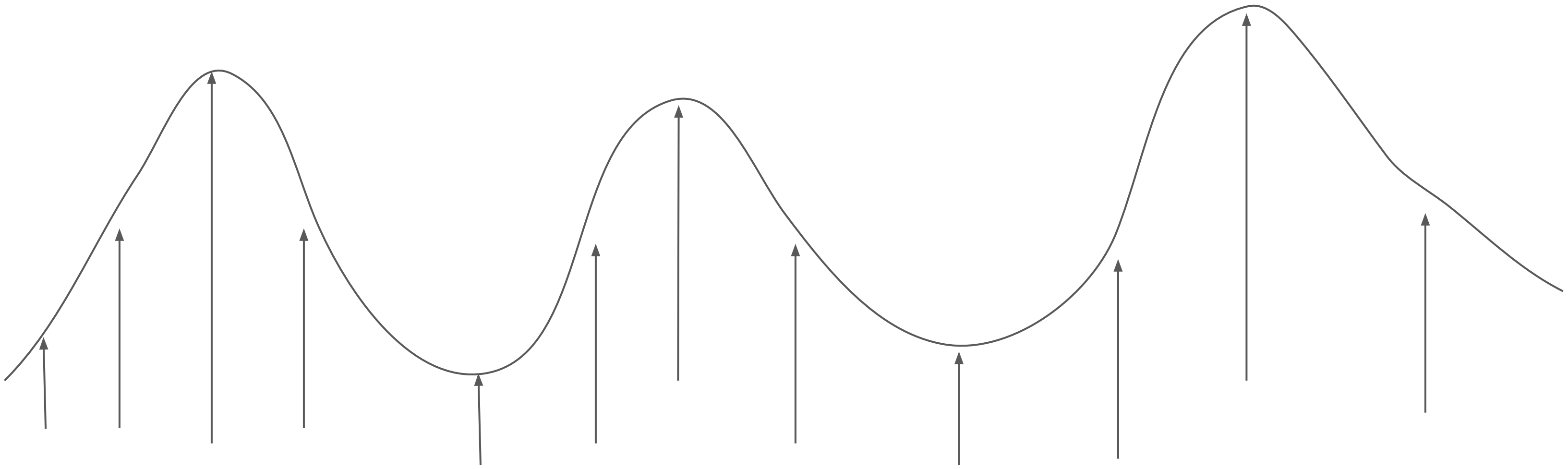
Example: A Weather Station



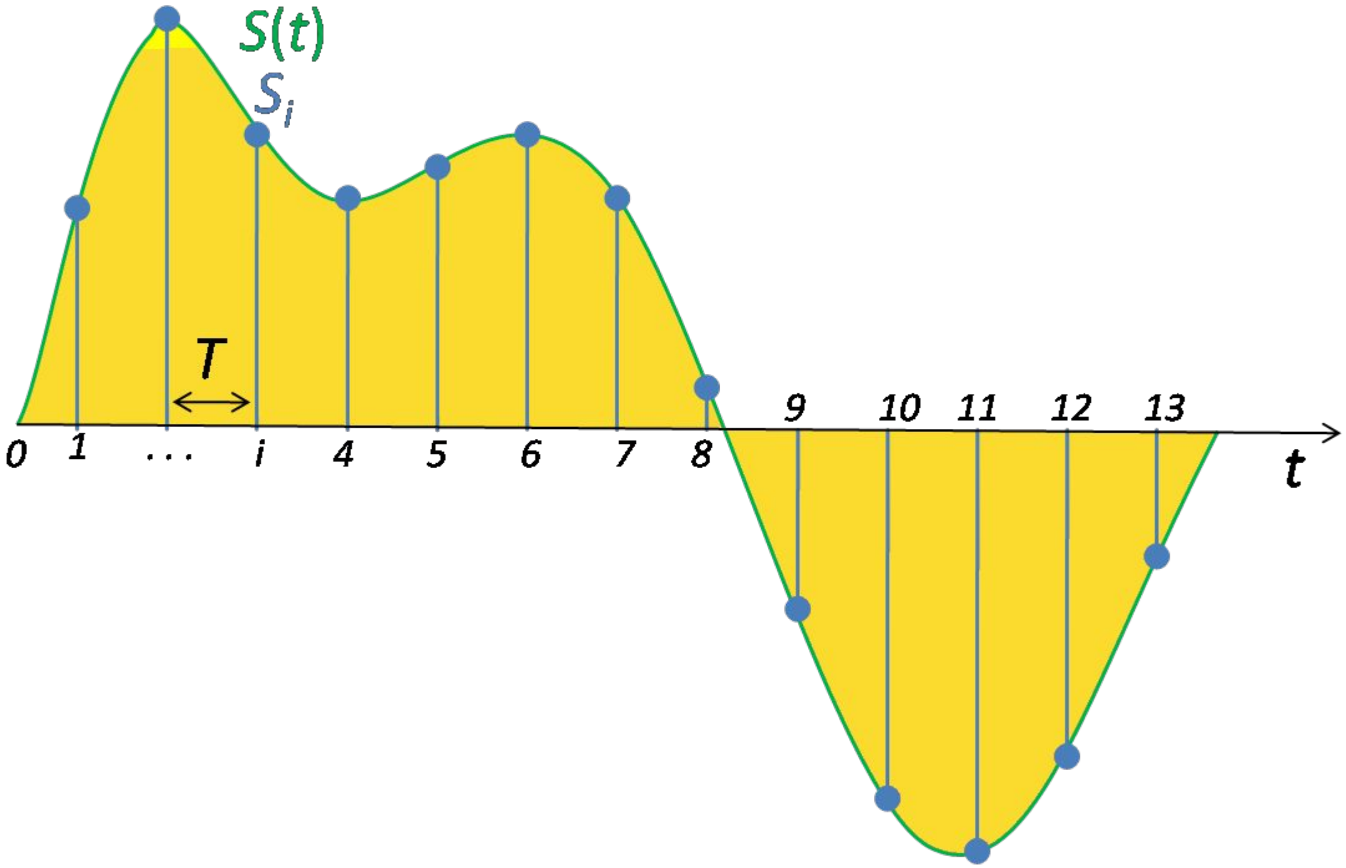
Example: A Weather Station



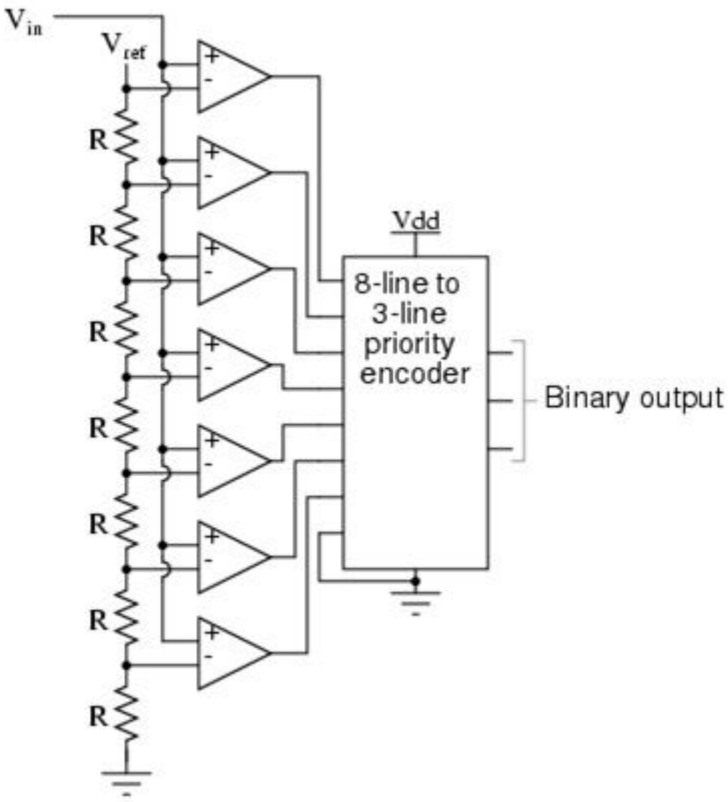
Example: A Weather Station



Example: A Weather Station



Time Quantization



The Analog to Digital Convertor (ADC)

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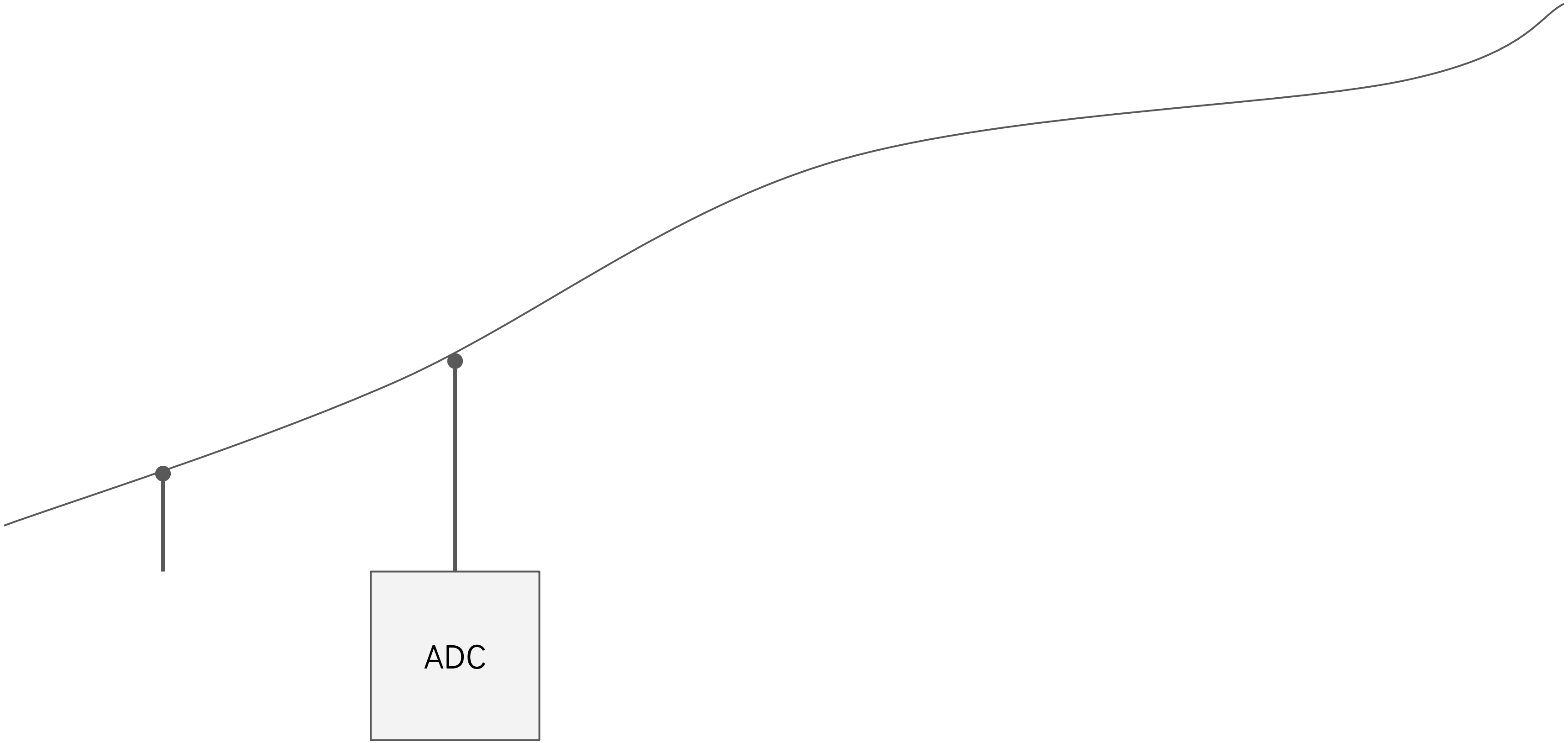
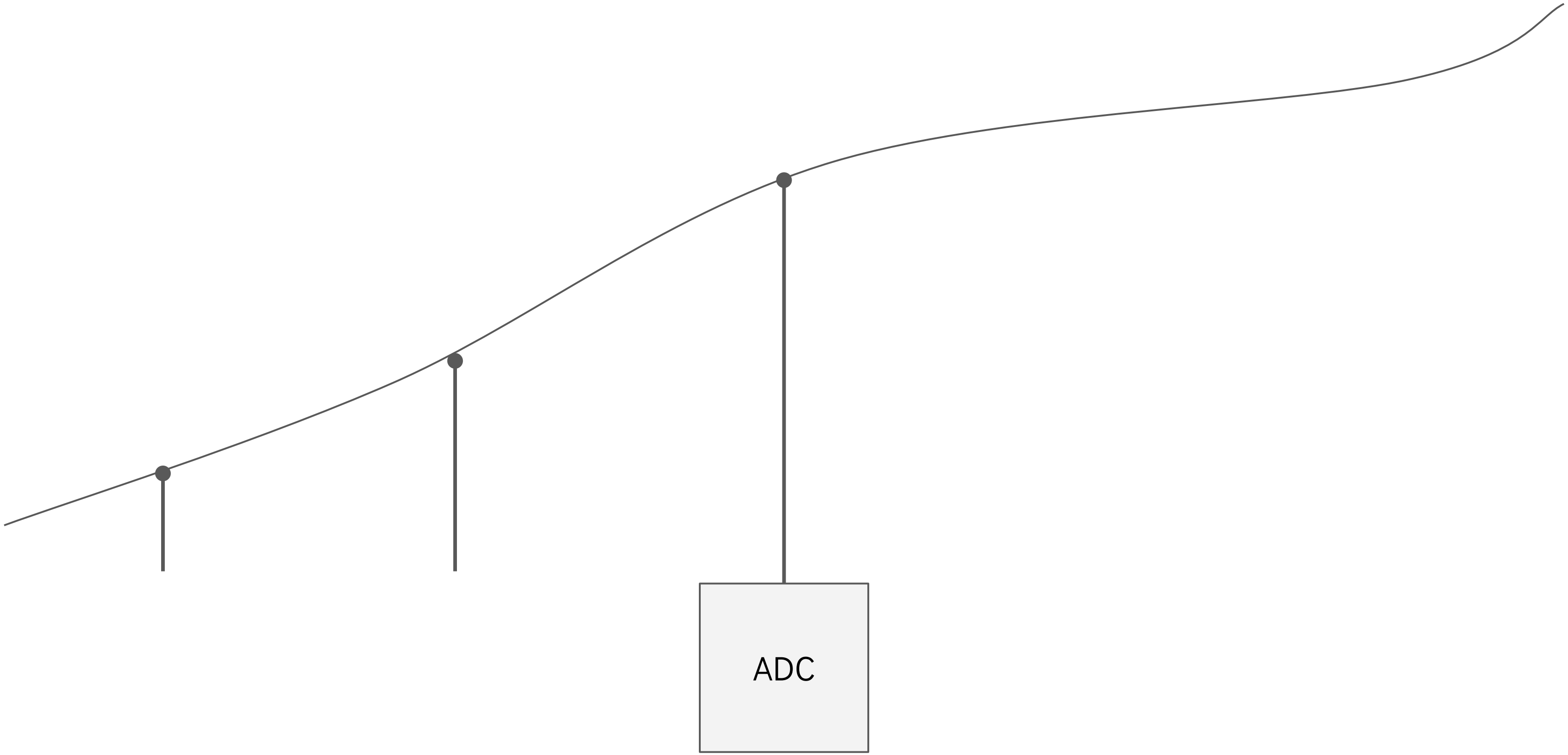
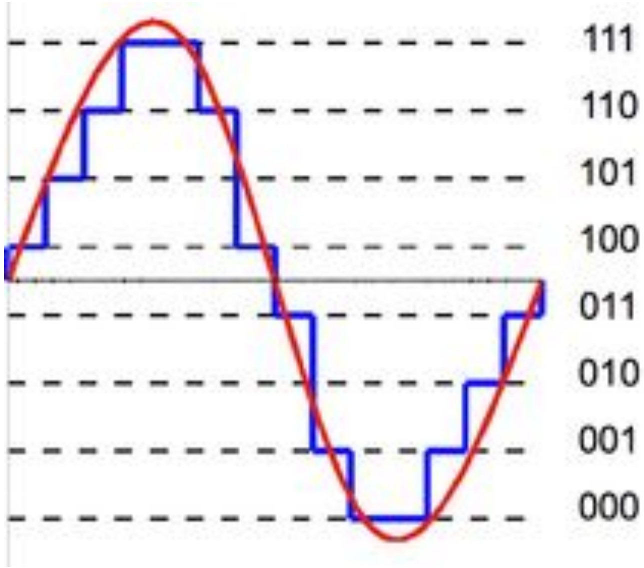


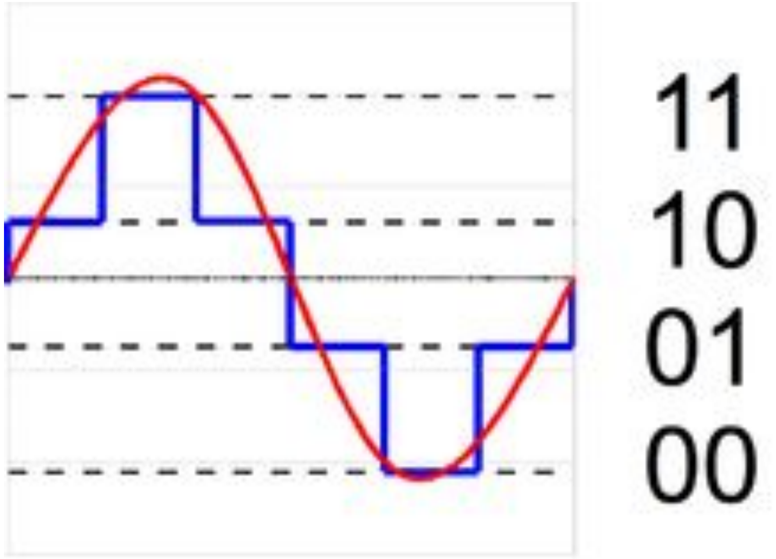
The Analog to Digital Convertor (ADC)

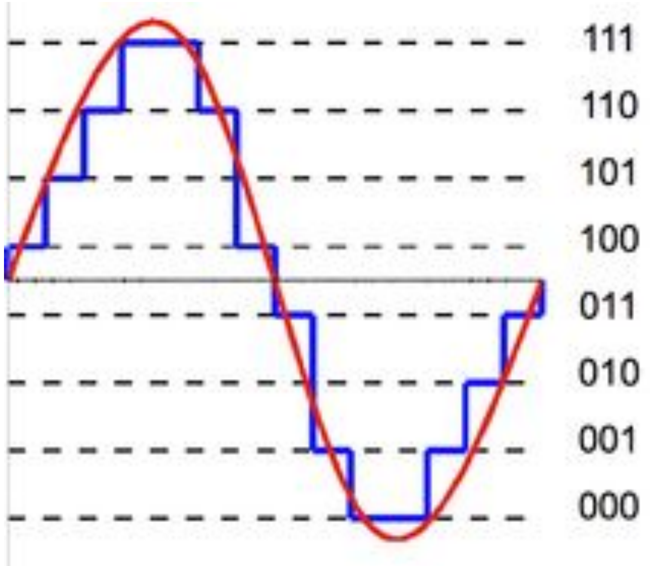


The Analog to Digital Convertor (ADC)

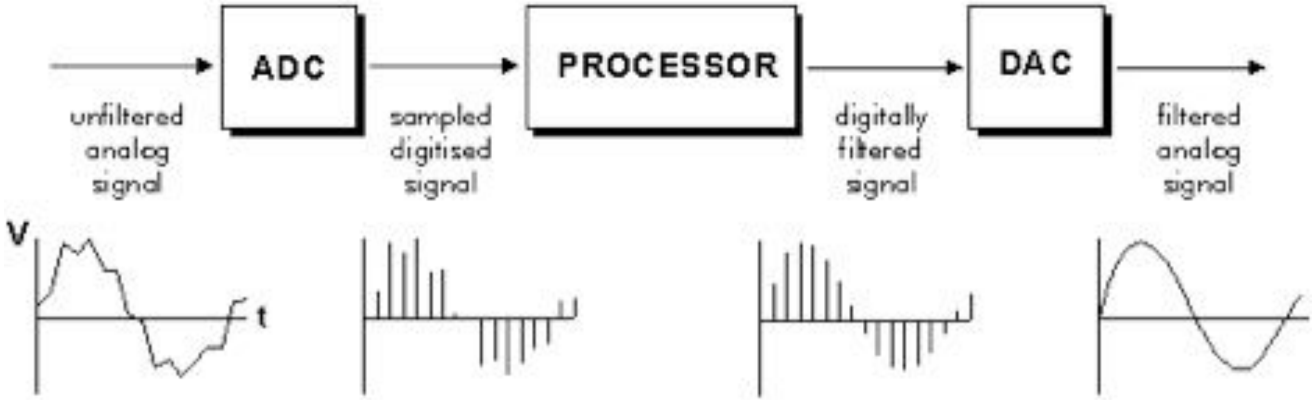


Amplitude Quantization

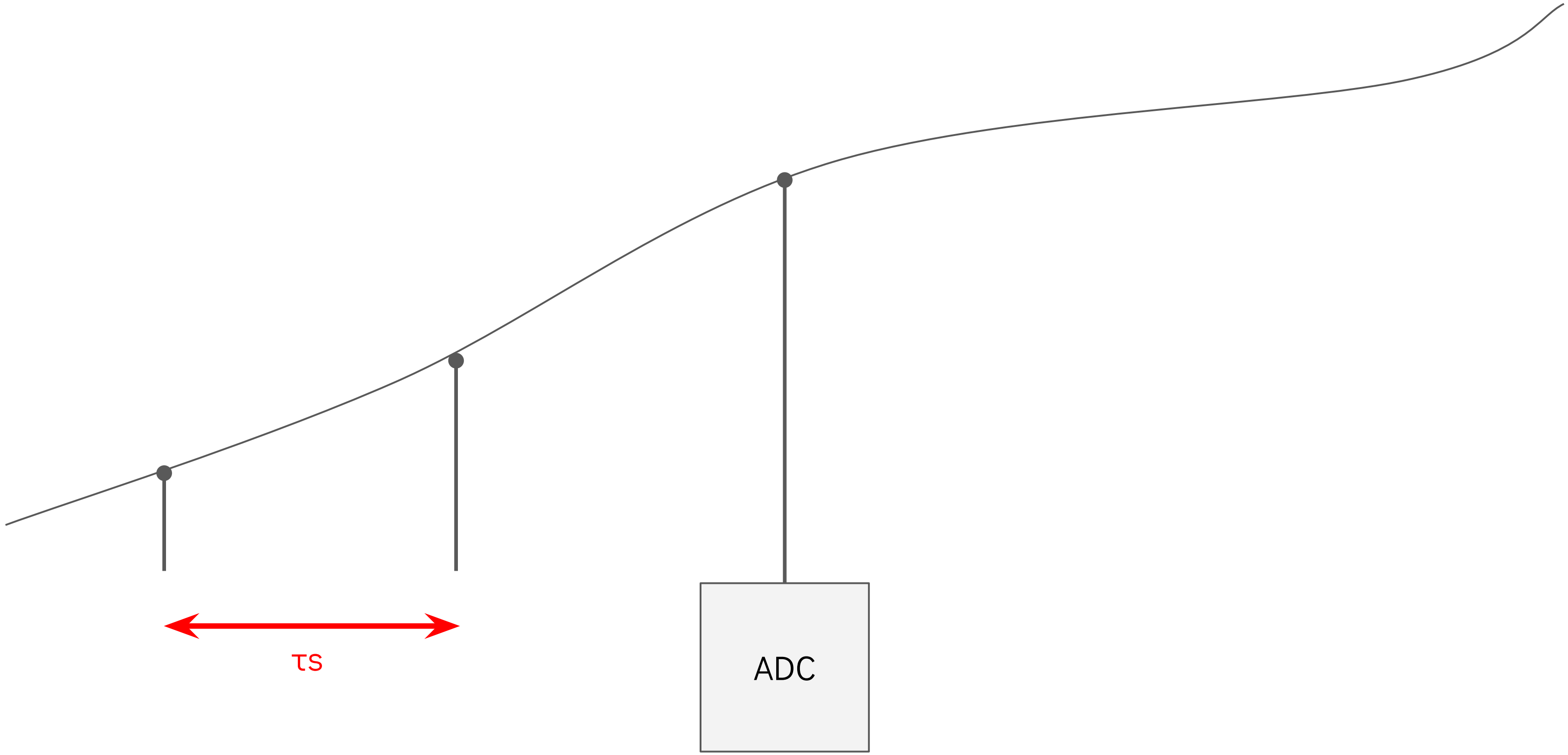




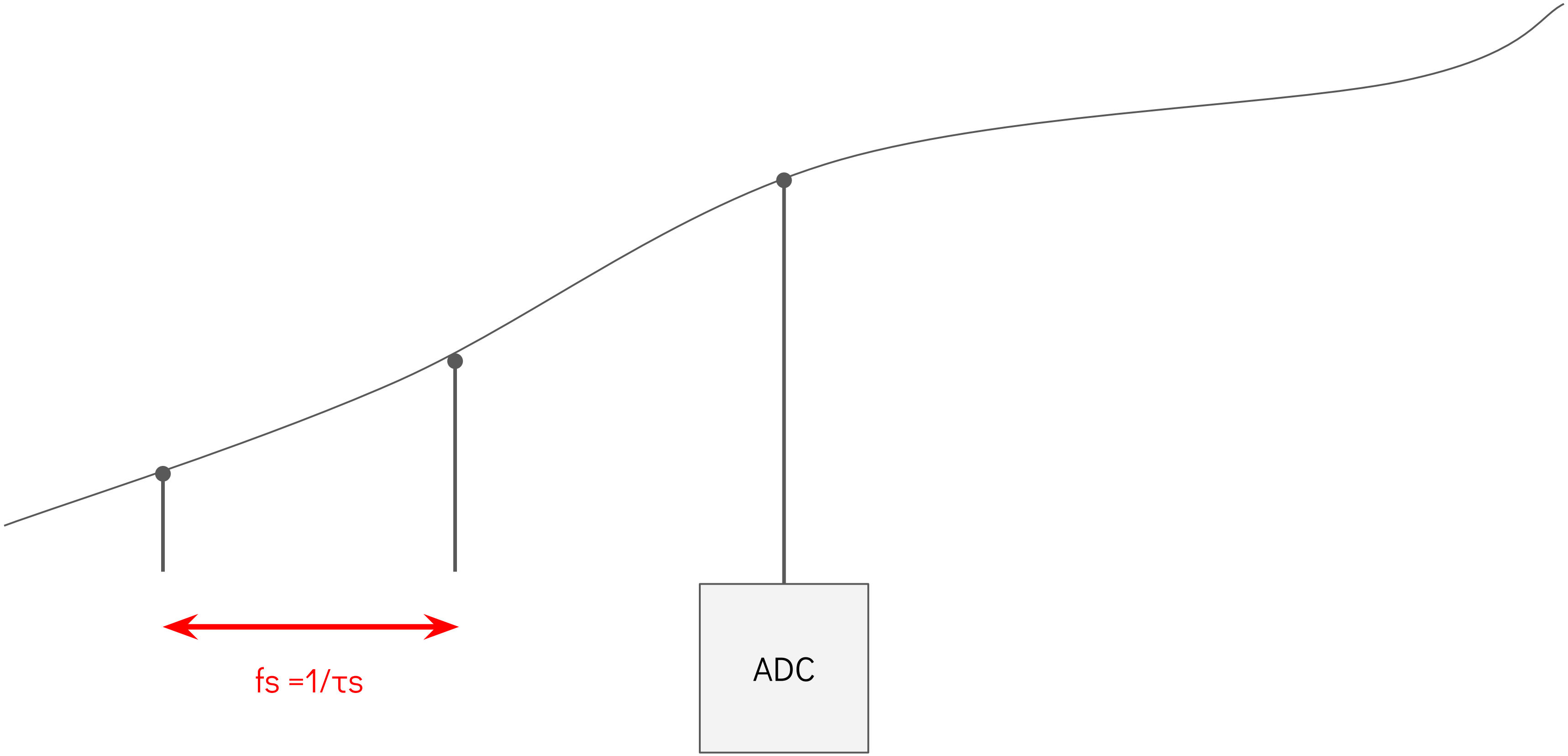
Sample Bit Depth



The DSP Process

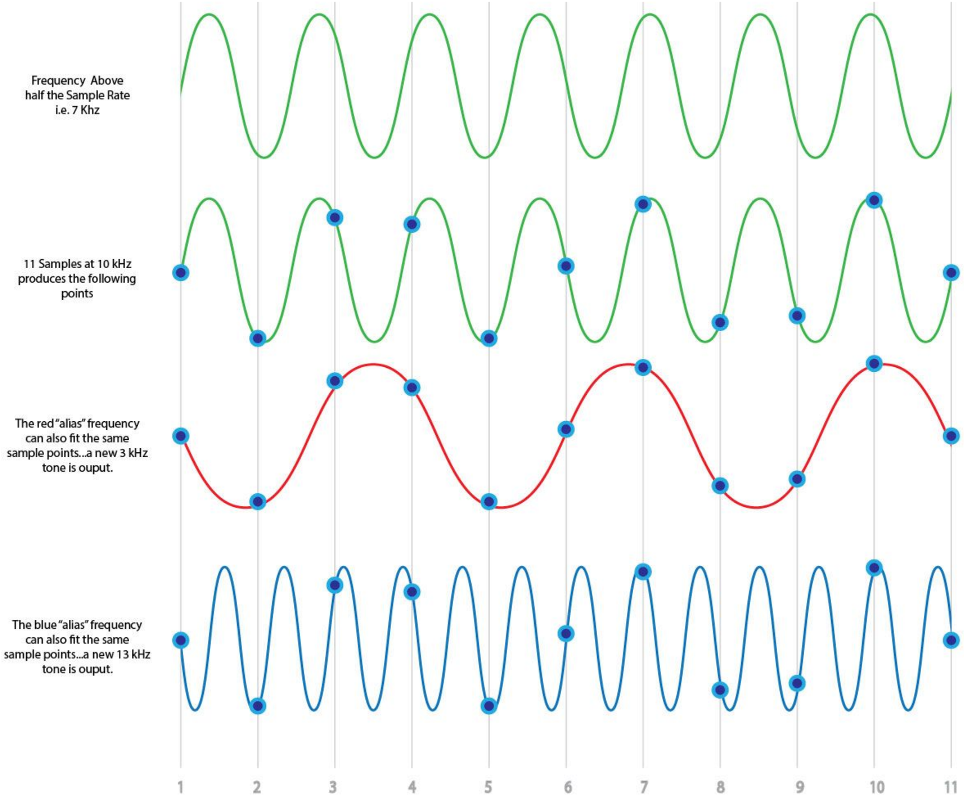


The Sampling Period



The Sample Rate

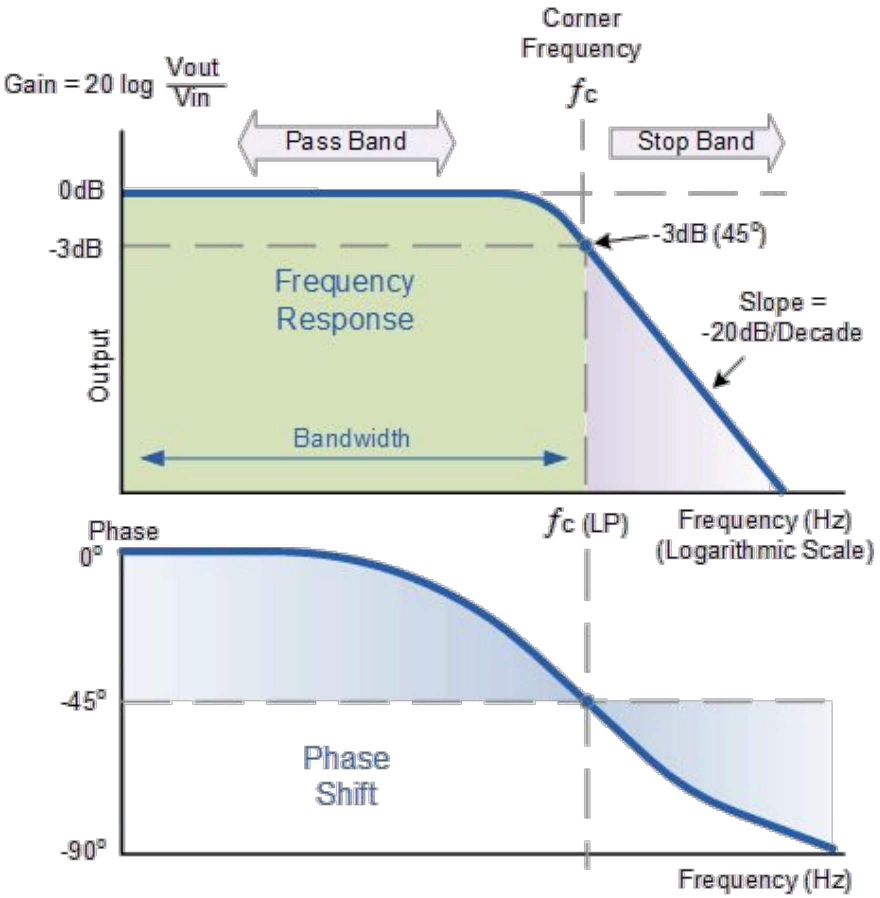
Aliasing



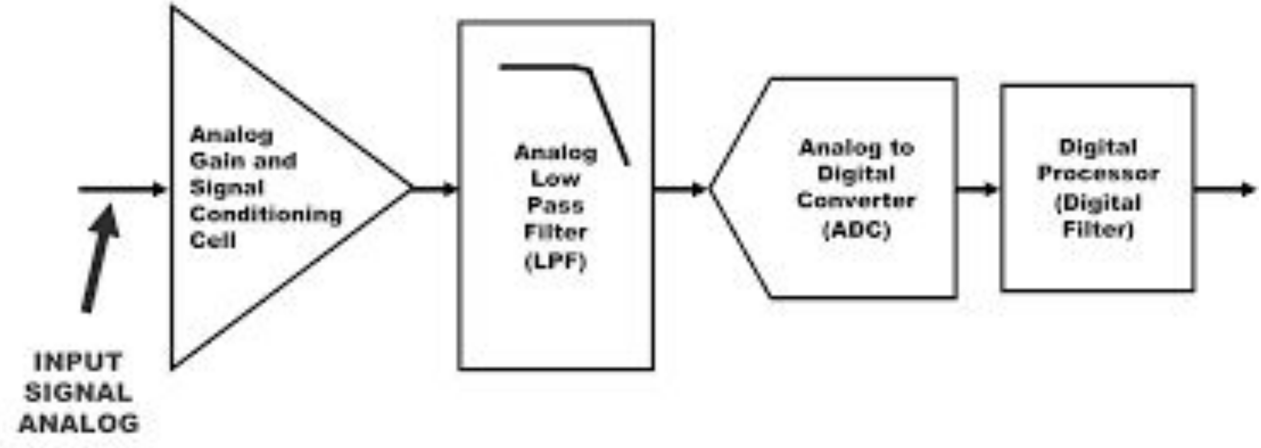
Aliasing

$$f_s > 2B$$

The Nyquist Rate

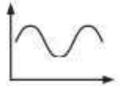
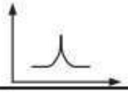
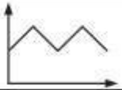
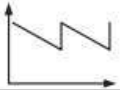
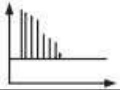
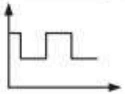
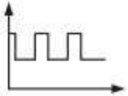
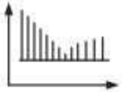
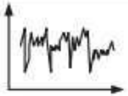
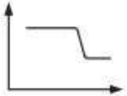
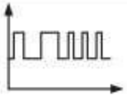


The Nyquist Rate

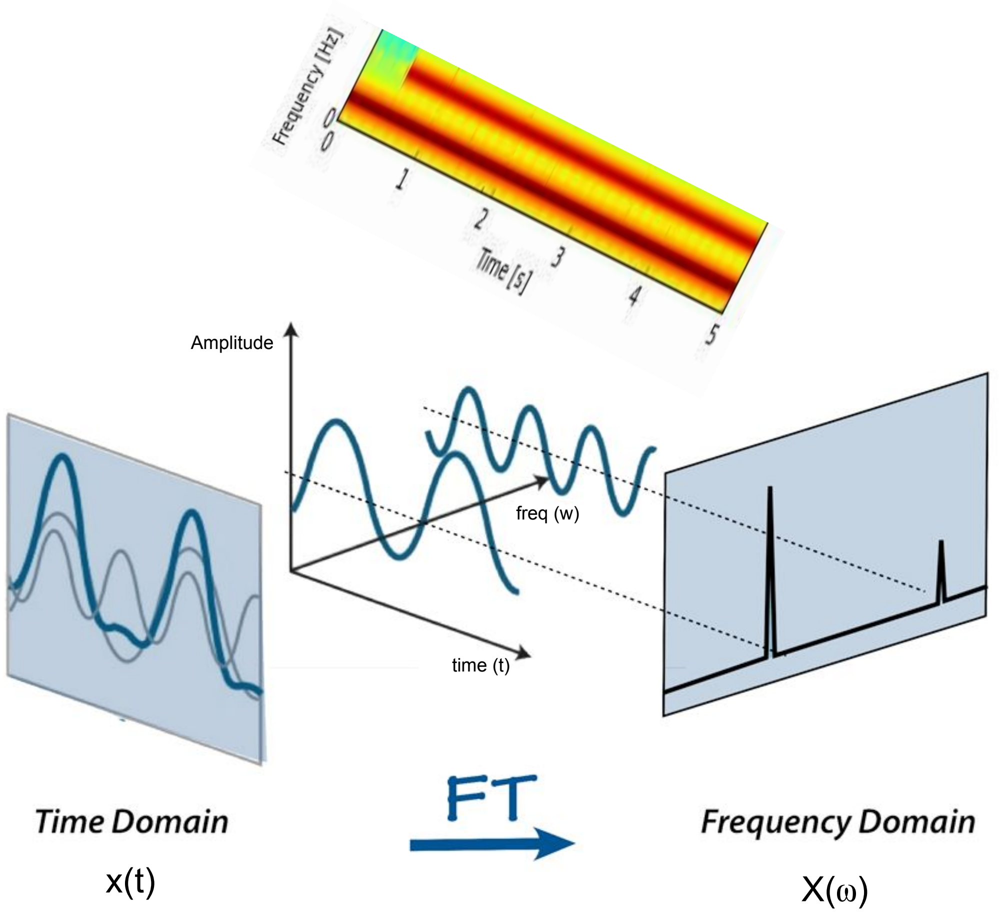


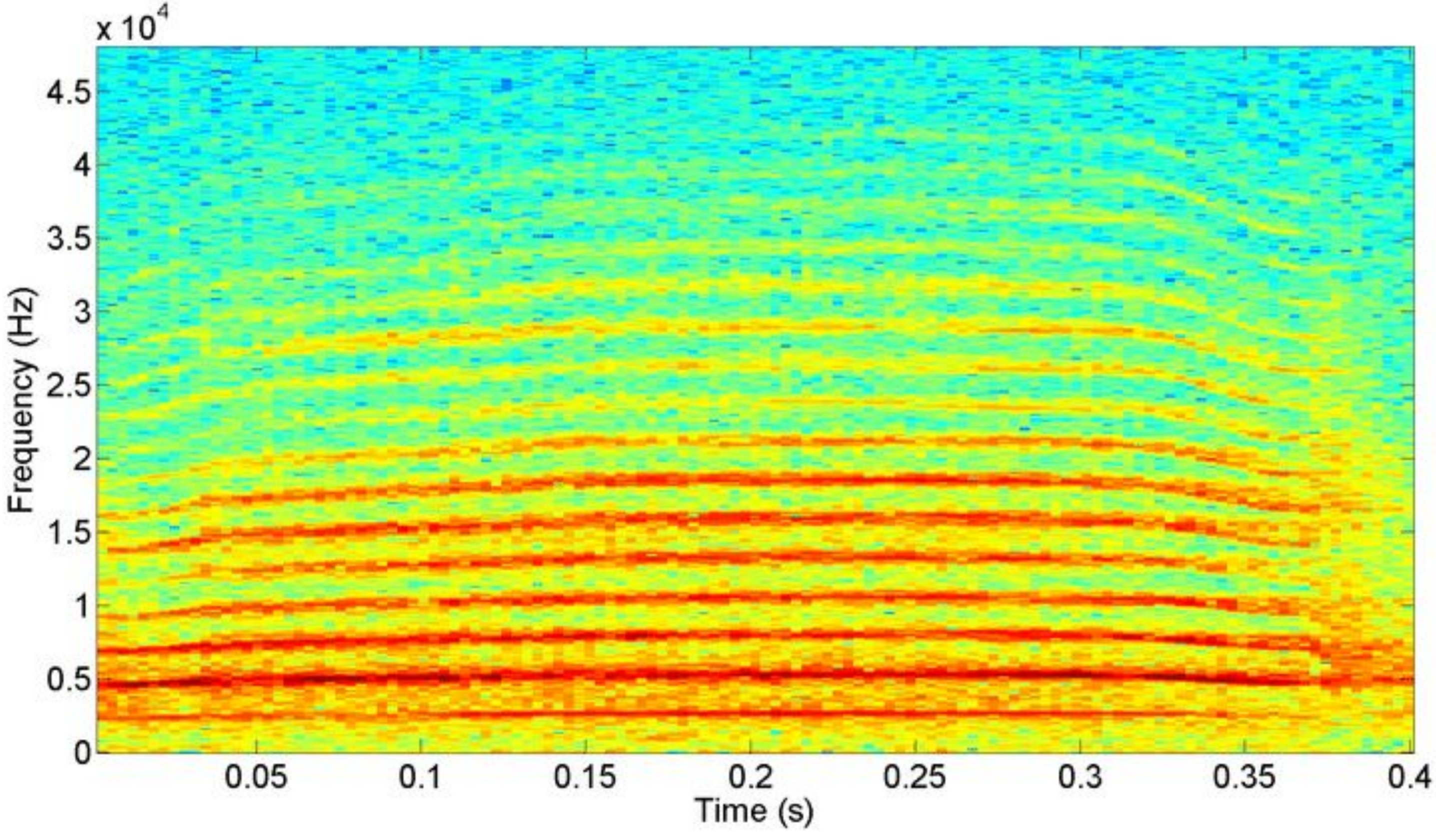
The Nyquist Rate

5.
THE FREQUENCY DOMAIN

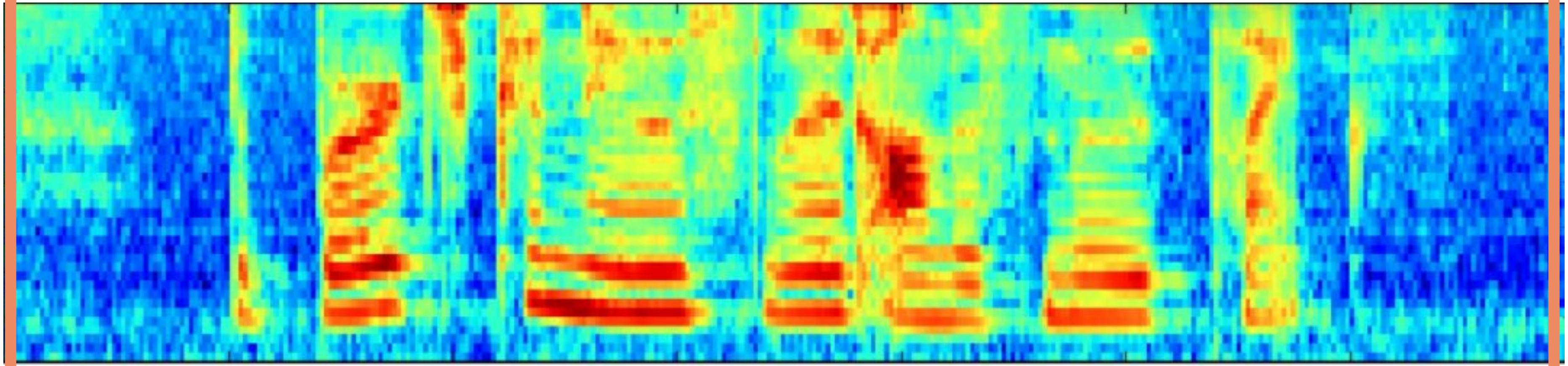
Waveform	Time domain	Frequency domain
Sinewave		
Triangle		
Sawtooth		
Rectangle		
Pulse		
Random noise		
Bandlimited noise		
Random binary sequence		

The Time and Frequency Domain


Frequency [Hz]
0
1
2
Time [s]
3
4
5
Amplitude
freq (w)
time (t)
Time Domain
x(t)
FT
Frequency Domain
X(ω)



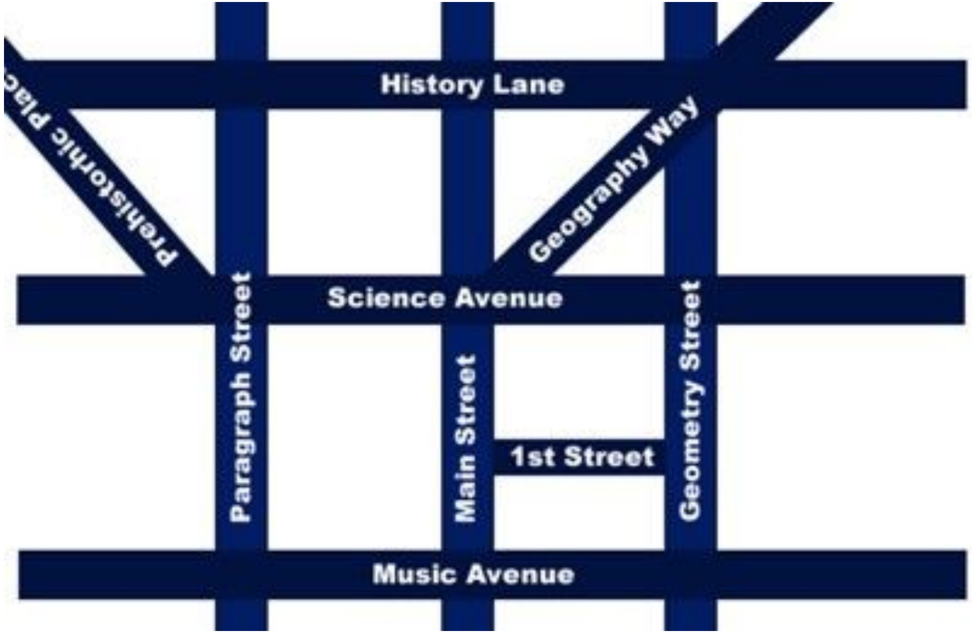
x 10^4
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Frequency (Hz)
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Time (s)



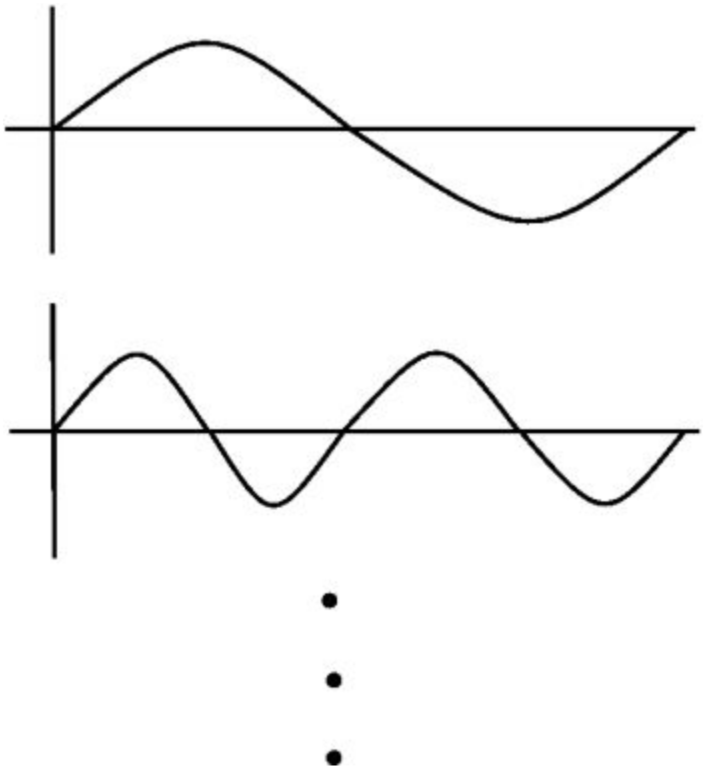
The Mel Spectrogram

And now…

Math



Orthogonality on a Map



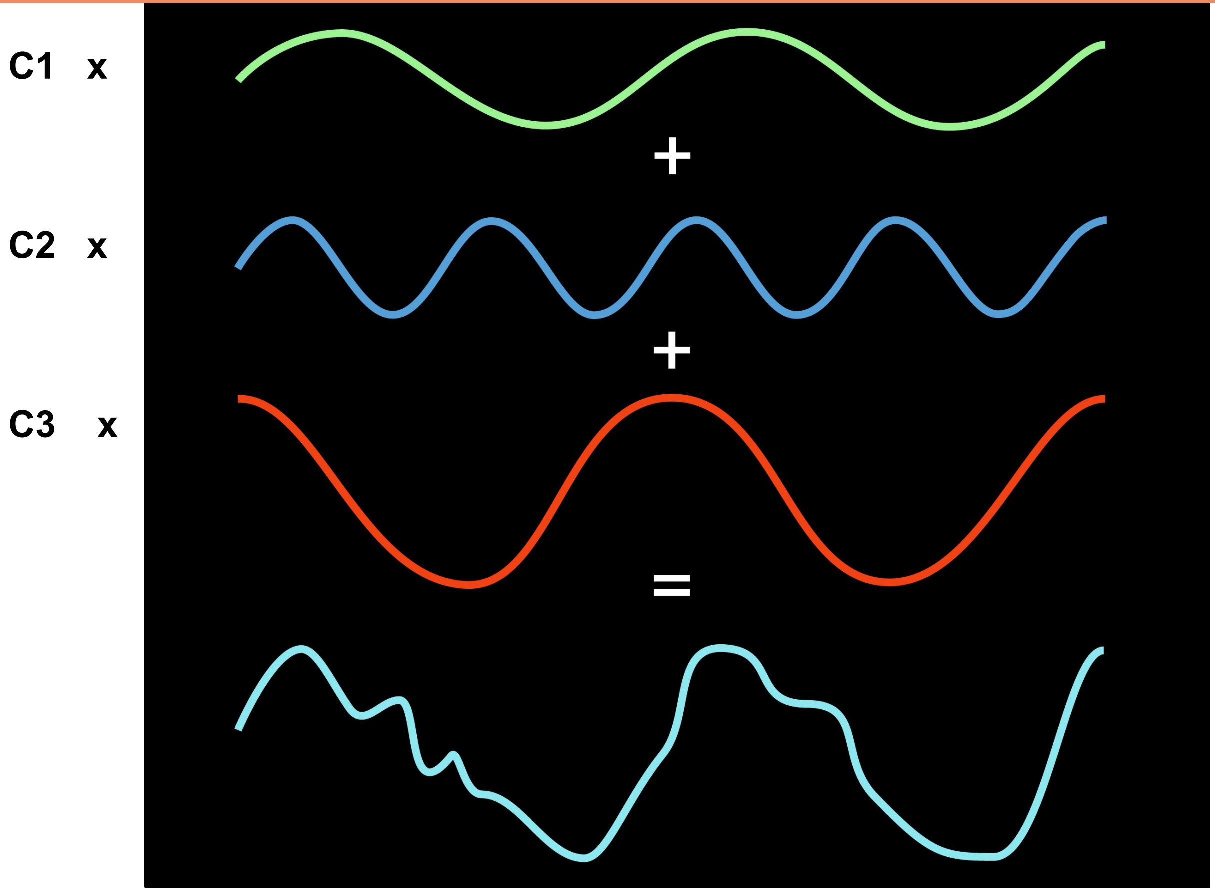
Orthogonality of Functions

$$\langle f, g \rangle = \int \overline{f(x)} g(x)\, dx.$$

Orthogonality of Functions

$$\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x)\, dx = \pi\delta_{mn}$$

Orthogonality of Functions

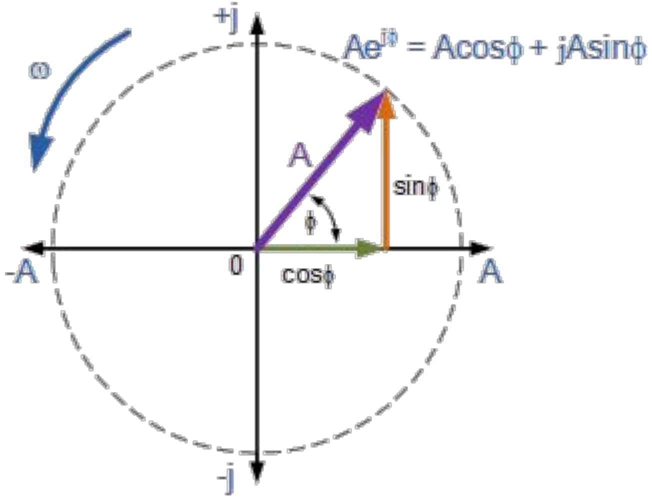


The Fourier Transform

$$f(t) = \int_0^\infty \left(a(\lambda)\cos(2\pi\lambda t) + b(\lambda)\sin(2\pi\lambda t)\right) d\lambda.$$

The Fourier Transform

$$Ae^{j\theta}e^{j\omega t} = Ae^{j(\omega t+\theta)} = A\left(\cos(\omega t + \theta) + j\sin(\omega t + \theta)\right)$$

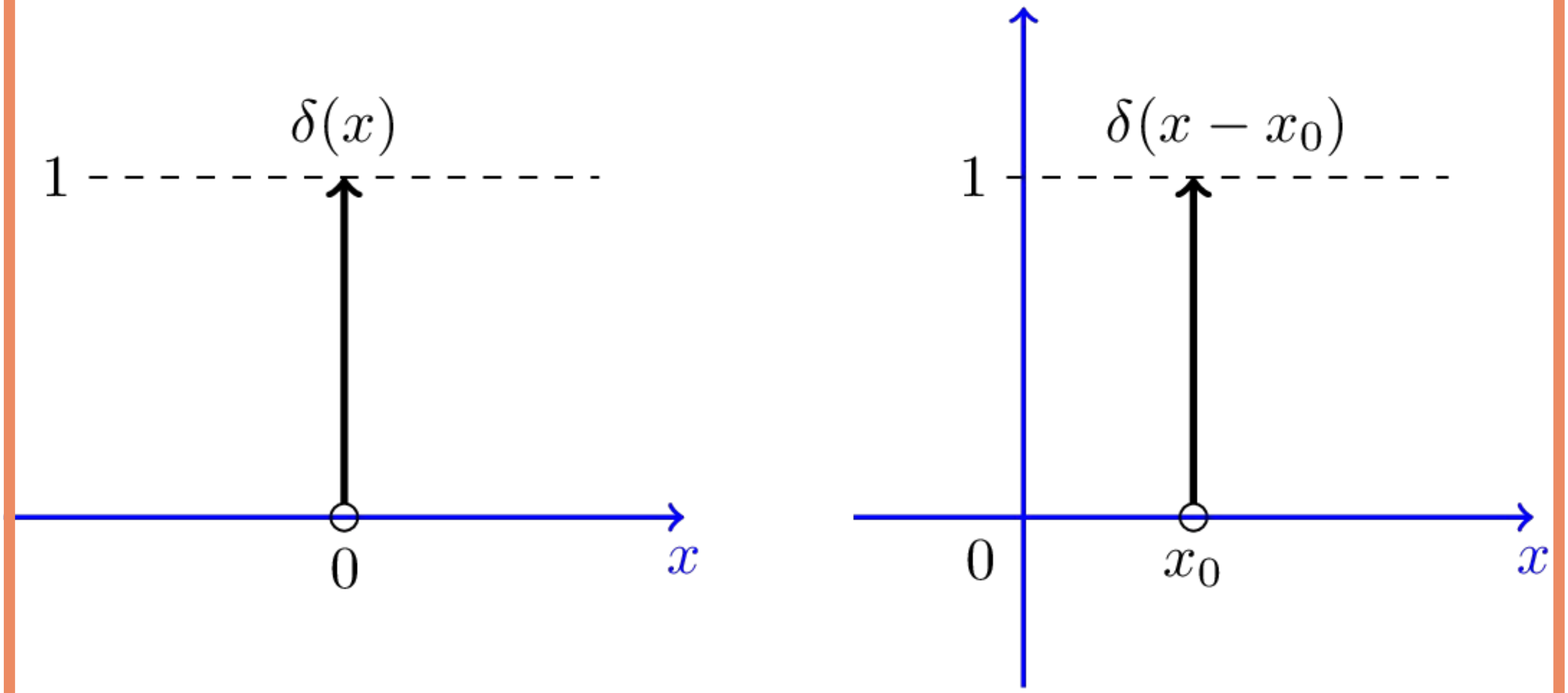


Complex Exponentials

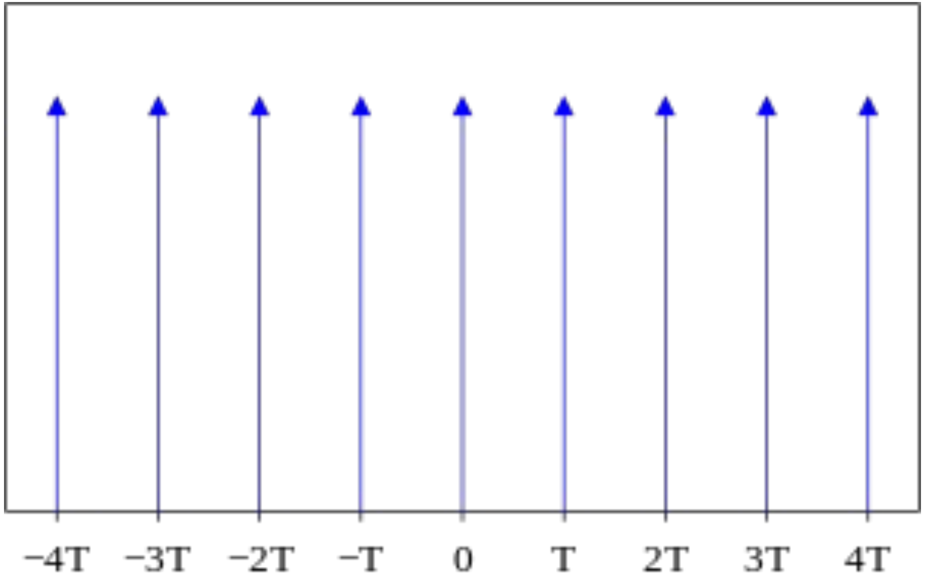
$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

The Fourier Transform and Inverse



Impulses



Impulses

Key Concept: Sifting Property of the Impulse

If a<b, then

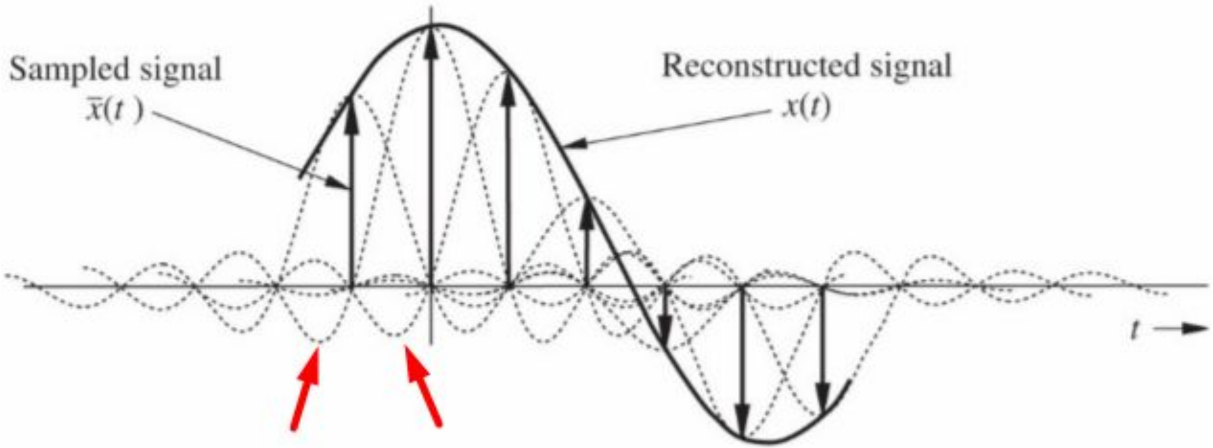
$$\int_a^b \delta(t-\lambda)\cdot f(t)dt = \begin{cases} f(\lambda), & a<\lambda<b \\ 0, & otherwise \end{cases}$$

↓

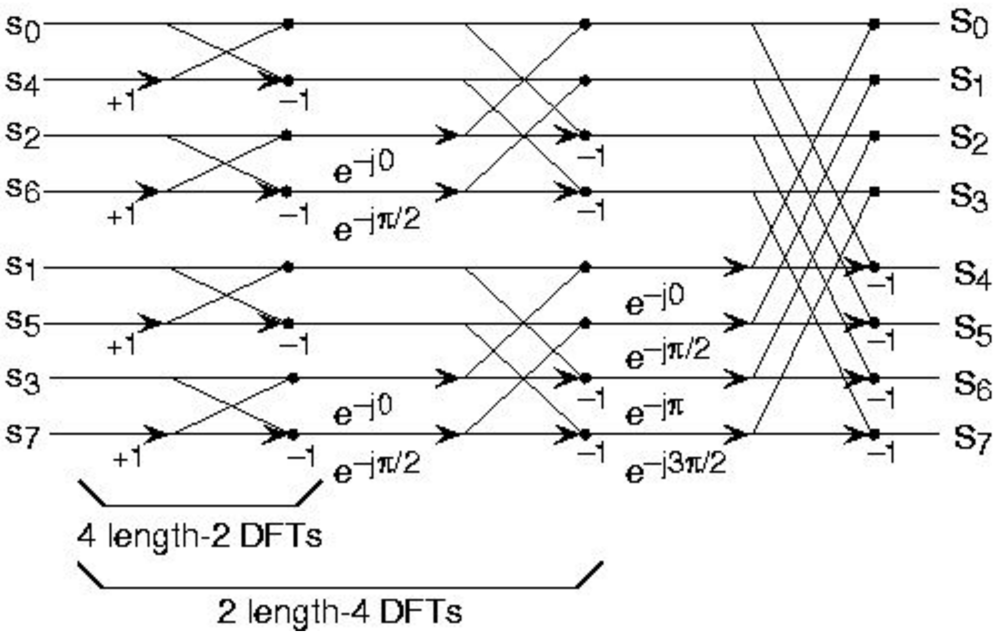
$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-jn\omega}$$

The Discrete Fourier Transform

$$x_r(t) = x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}(2\pi Bt - n\pi)$$



Impulses


s0
s4
s2
s6
s1
s5
s3
s7
+1
−1
e−j0
e−jπ/2
e−jπ
e−j3π/2
S0
S1
S2
S3
S4
S5
S6
S7
4 length-2 DFTs
2 length-4 DFTs

The FFT

```
RECURSIVE-FFT(a)
n ← length[a]                       ▷ n is a power of 2.
if n = 1
    then return a
ω_n ← e^{2πi/n}
ω ← 1
a^[0] ← (a_0, a_2, ..., a_{n-2})
a^[1] ← (a_1, a_3, ..., a_{n-1})
y^[0] ← RECURSIVE-FFT(a^[0])
y^[1] ← RECURSIVE-FFT(a^[1])
for k ← 0 to n/2 - 1
     do y_k ← y_k^[0] + ω y_k^[1]
        y_{k+(n/2)} ← y_k^[0] - ω y_k^[1]
        ω ← ω ω_n
return y                            ▷ y is assumed to be column vector.
```

The FFT

6.
INFORMATION THEORY

A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

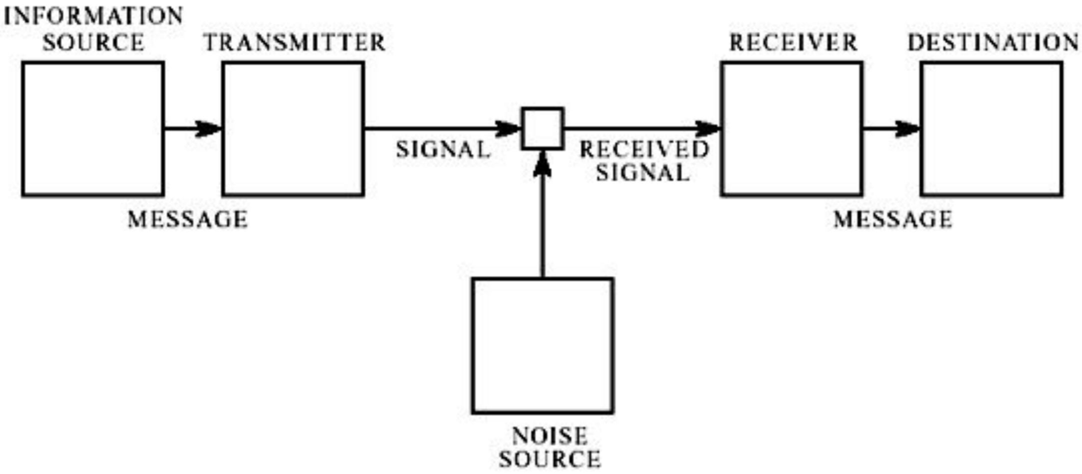
THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

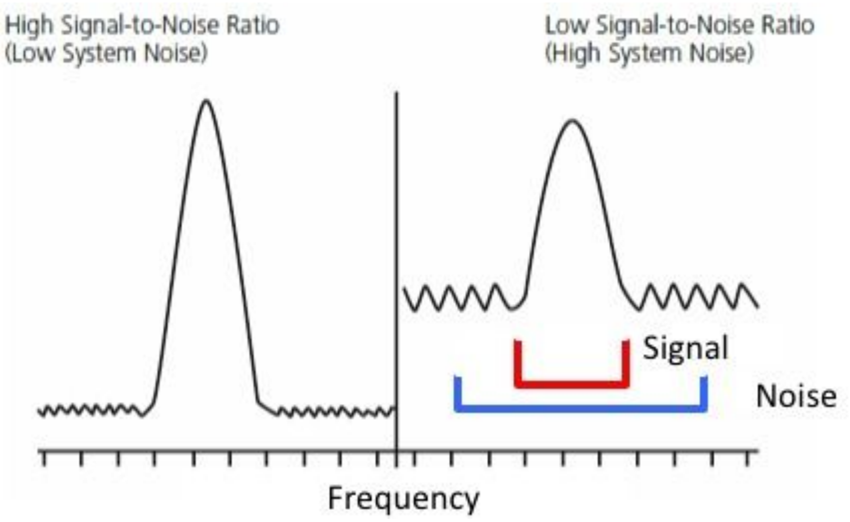
If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded as a measure of the information produced when one message is chosen from the set, all choices being equally likely. As was pointed out by Hartley the most natural choice is the logarithmic function. Although this definition must be generalized considerably when we consider the influence of the statistics of the message and when we have a continuous range of messages, we will in all cases use an essentially logarithmic measure.

The logarithmic measure is more convenient for various reasons:

Origins of Information Theory



The Channel Model



Signal to Noise Ratio (SNR)

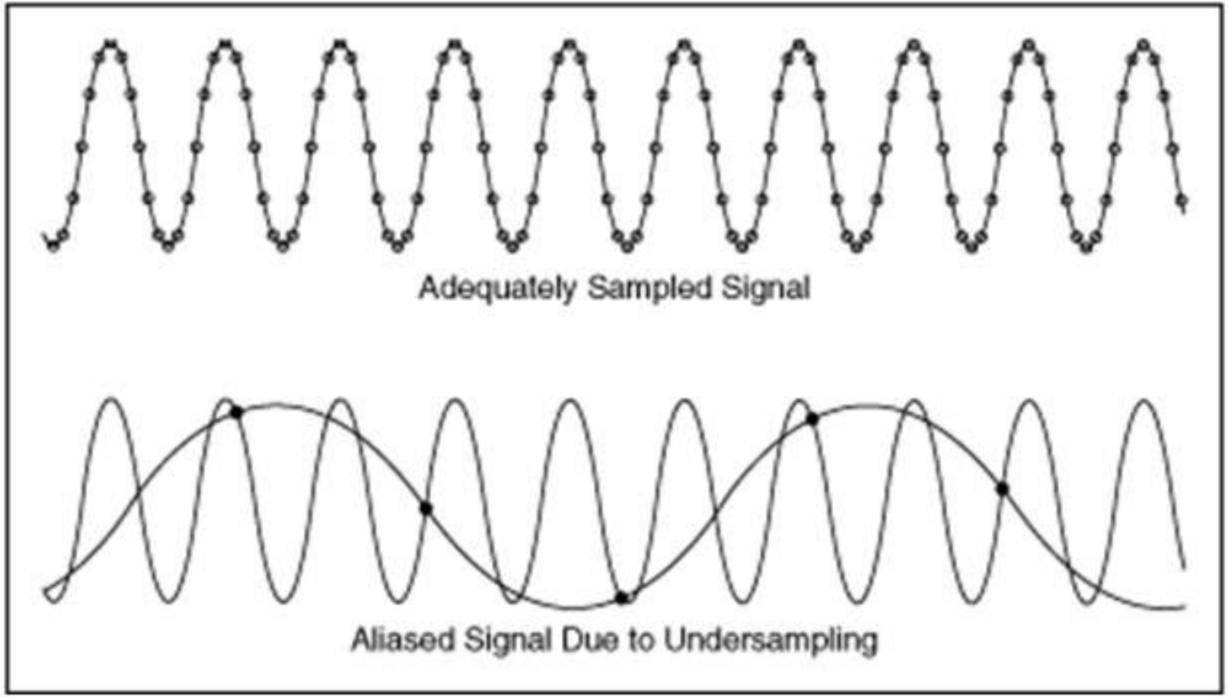
$$H(X) = -\sum_{i=1}^{n} p_i \log_2 p_i$$

Information Entropy

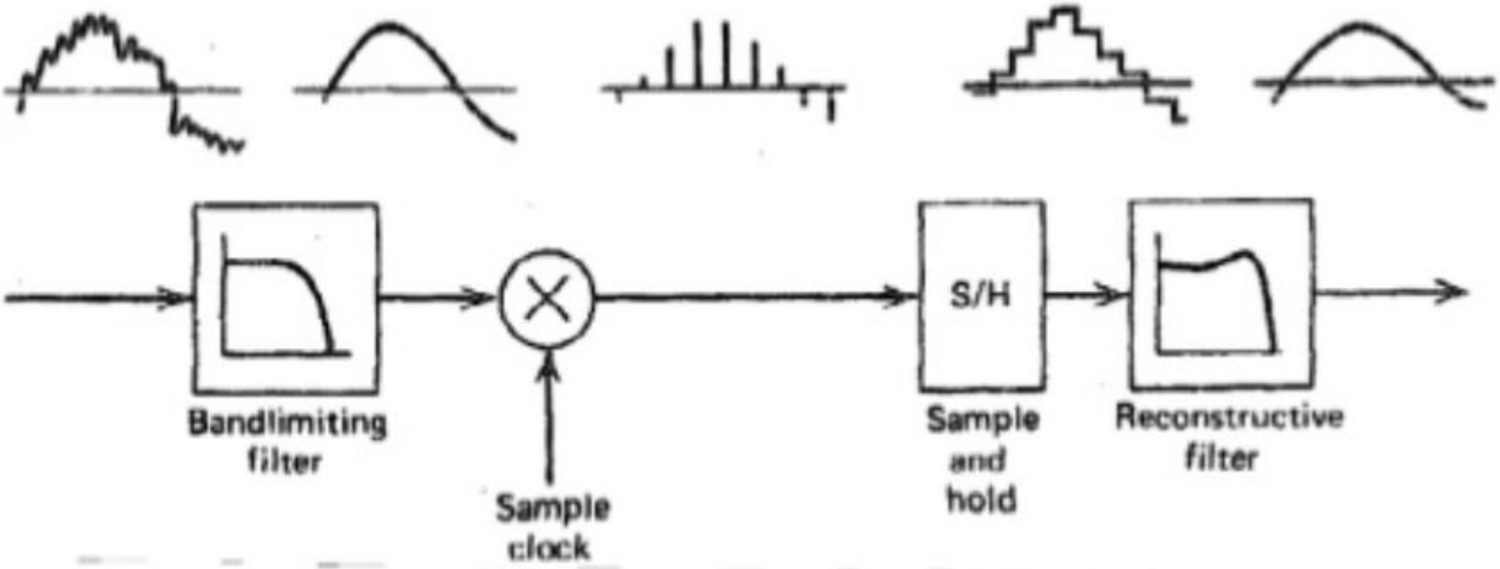
$$I < B \log_2\left(1 + \frac{S}{N}\right)$$

Maximum Rate or
Channel Capacity
(bits per second)

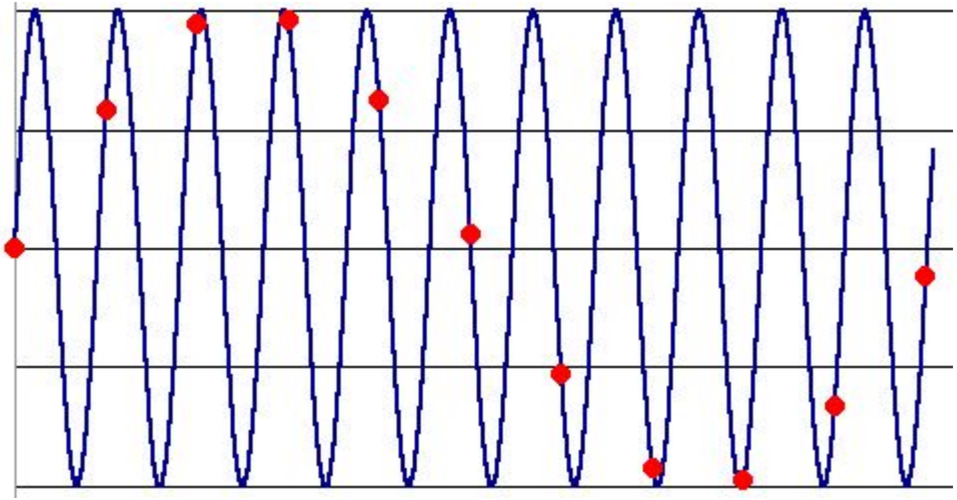
Shannon–Hartley Theorem



Nyquist–Shannon sampling theorem



Band Limiting

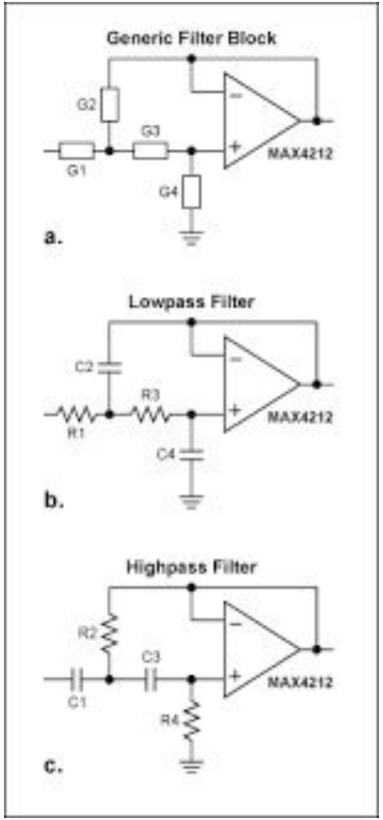


Aliasing and Images

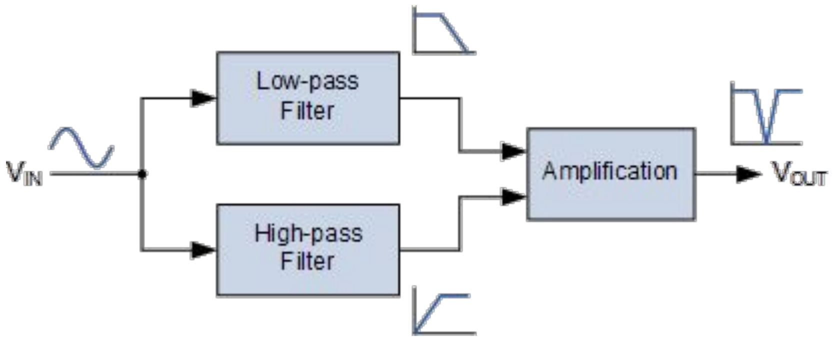
Processing

7.

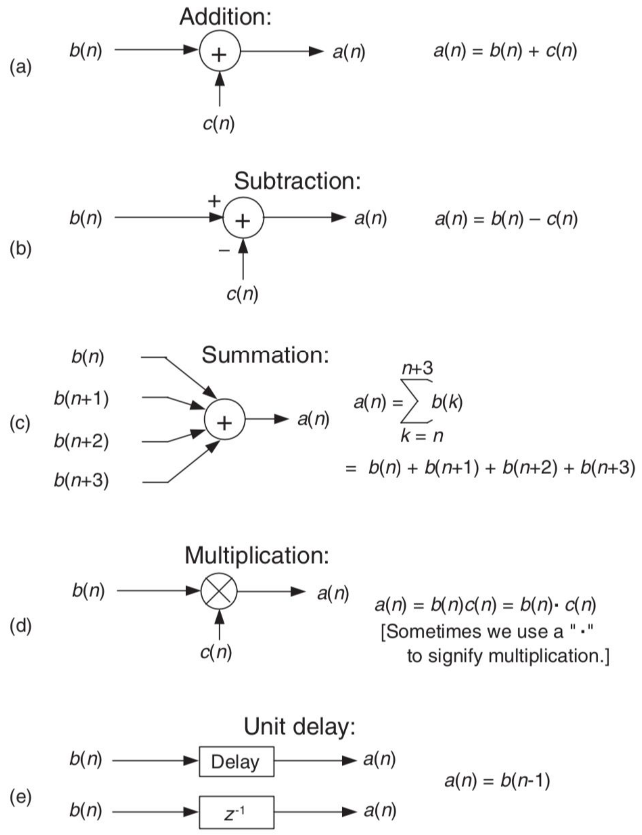
FILTERING



Analog Filters



Analog Filter Block Diagram



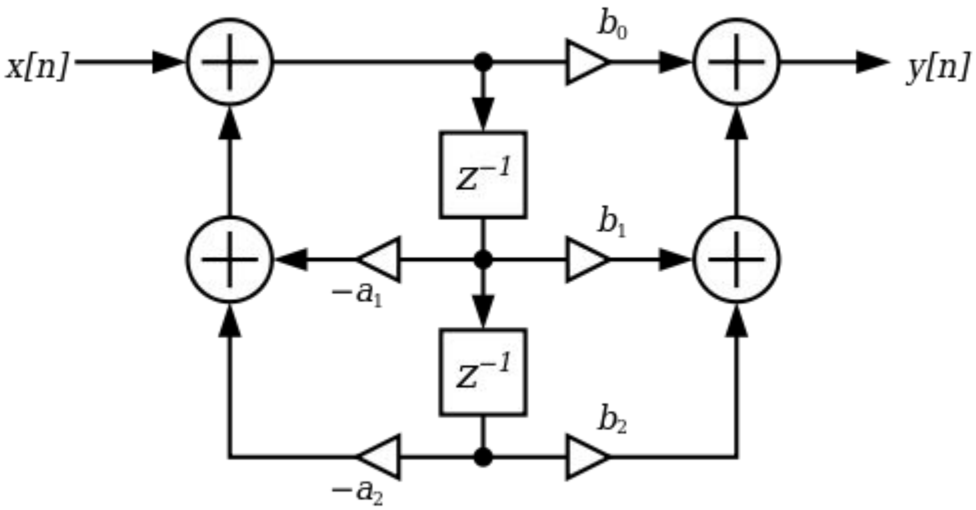
Digital Filter Block Diagram

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-jn\omega}$$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Z-Transform

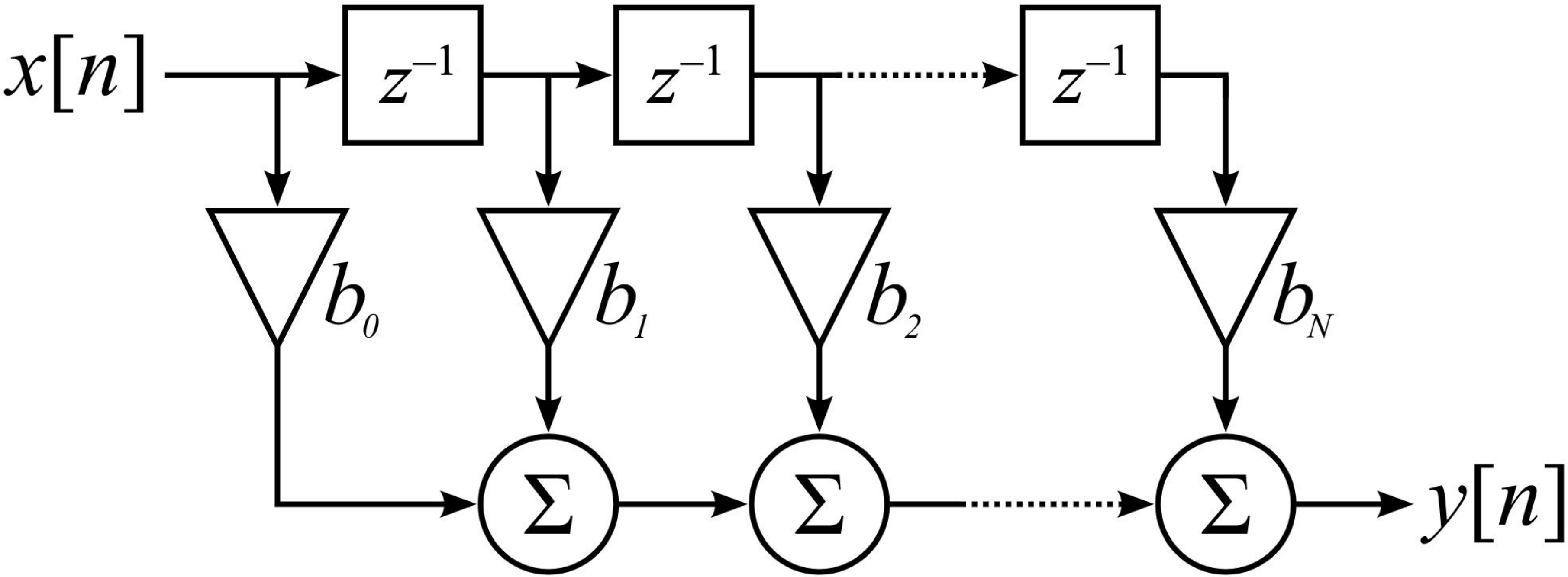
The Z Transform



IIR Filters

$$y[n] = \frac{1}{a_0}(b_0 x[n] + b_1 x[n-1] + \cdots + b_P x[n-P] - a_1 y[n-1] - a_2 y[n-2] - \cdots - a_Q y[n-Q])$$

IIR Filters



FIR Filters

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$
$$= \sum_{i=0}^{N} b_i \cdot x[n-i],$$

FIR Filters

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-jn\omega}$$

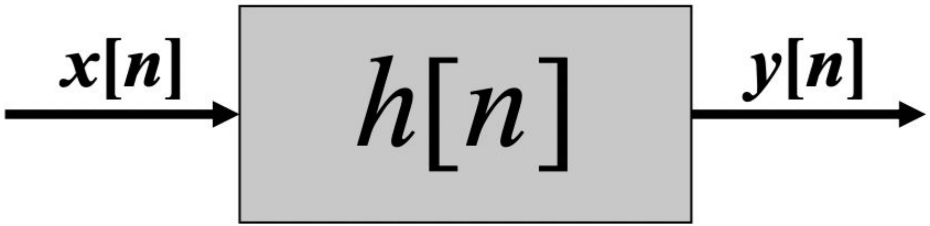
$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

The Z-Transform

Z Transform Pairs		
Time Domain *	**Z Domain**	
	z	**z^{-1}**
δ[k] (unit impulse)	1	1
γ[k] † (unit step)	$\Gamma(z)=\frac{z}{z-1}$	$\Gamma(z)=\frac{1}{1-z^{-1}}$
a^k	$\frac{z}{z-a}$	$\frac{1}{1-z^{-1}a}$
e^{-bTk}	$\frac{z}{z-e^{-bT}}$	$\frac{1}{1-z^{-1}e^{-bT}}$
k	$\frac{z}{(z-1)^2}$	$\frac{z^{-1}}{(1-z^{-1})^2}$
sin(bk)	$\frac{z\sin(b)}{z^2-2z\cos(b)+1}$	$\frac{z^{-1}\sin(b)}{1-2z^{-1}\cos(b)+z^{-2}}$
cos(bk)	$\frac{z(z-\cos(b))}{z^2-2z\cos(b)+1}$	$\frac{1-z^{-1}\cos(b)}{1-2z^{-1}\cos(b)+z^{-2}}$
$a^k\sin(bk)$	$\frac{az\sin(b)}{z^2-2az\cos(b)+a^2}$	$\frac{az^{-1}\sin(b)}{1-2az^{-1}\cos(b)+a^2z^{-2}}$
$a^k\cos(bk)$	$\frac{z(z-a\cos(b))}{z^2-2az\cos(b)+a^2}$	$\frac{1-az^{-1}\cos(b)}{1-2az^{-1}\cos(b)+a^2z^{-2}}$

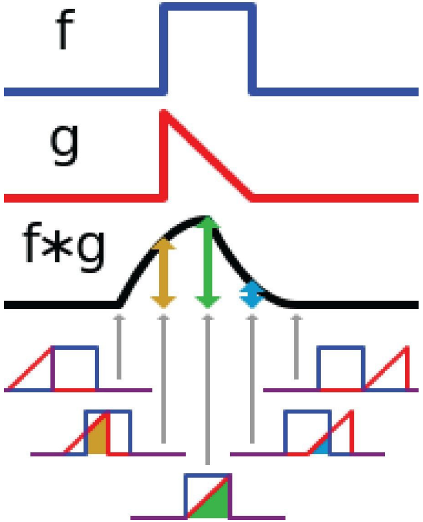
The Z-Transform

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

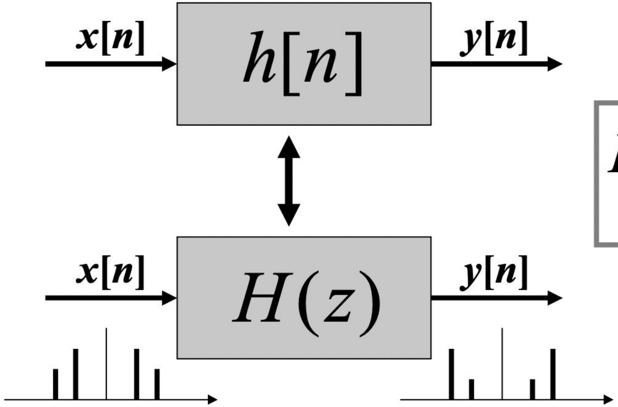


The Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$



The Convolution

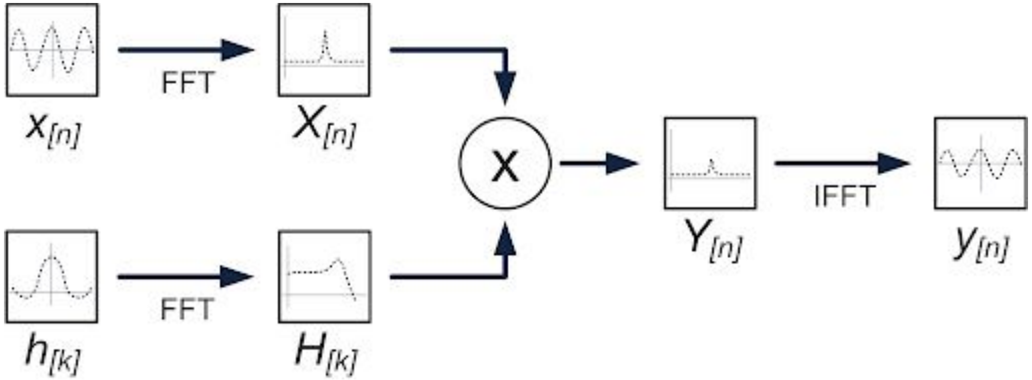


$$H(z) = \sum_n h[n] z^{-n}$$

$$y[n] = h[n] * x[n] \overset{z}{\longleftrightarrow} Y(z) = H(z)X(z)$$

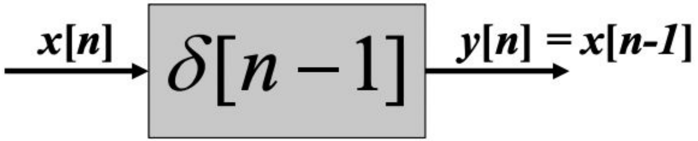
$$H(z) \equiv \frac{Y(z)}{X(z)}$$

The Transfer Function



$$H(z) \equiv \frac{Y(z)}{X(z)}$$

Convolution and the FFT



$$Y(z) = z^{-1} X(z)$$

Unit Delay

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$
$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

↓

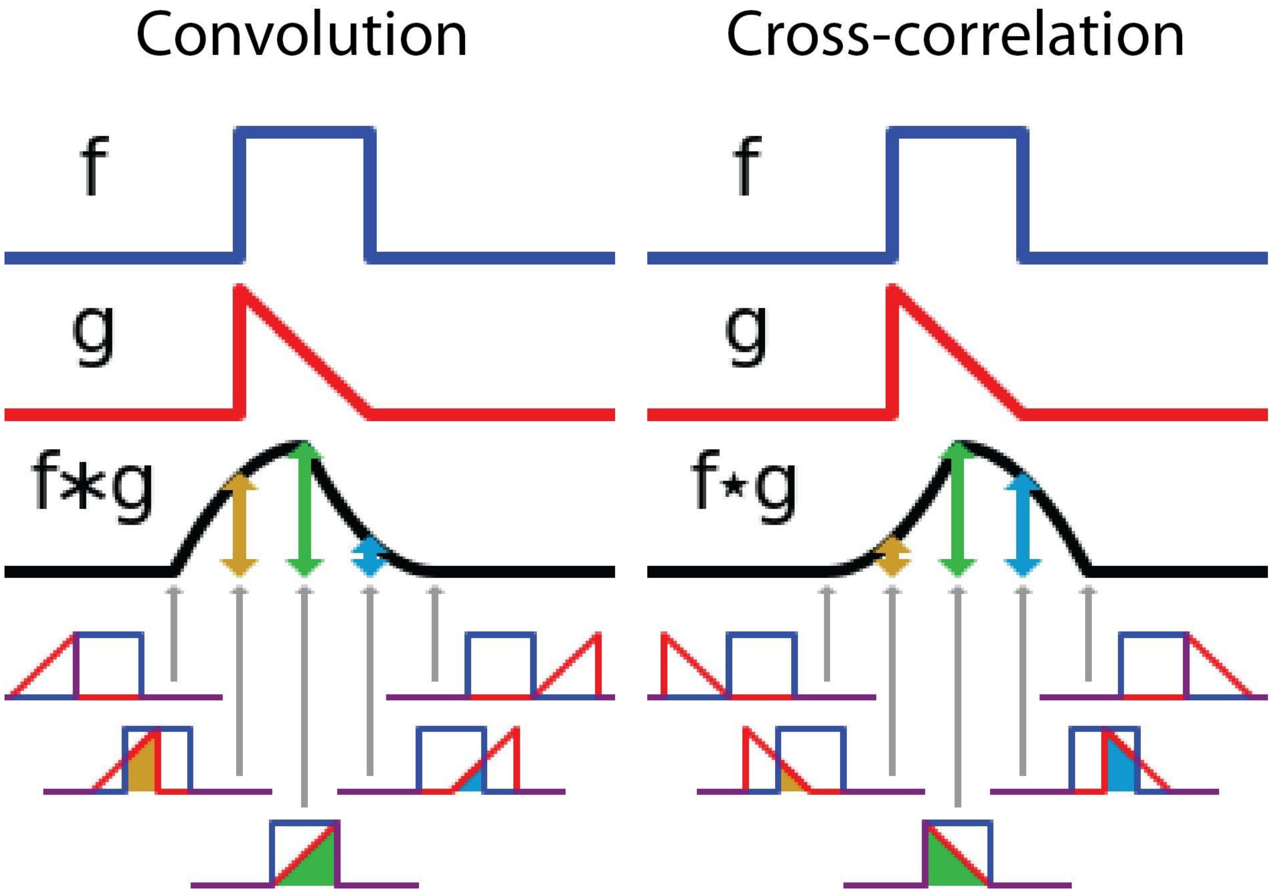
$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$
$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

↓

$$Y(z) = H(z)X(z) =$$
$$(0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4})(1 + 2z^{-1} + 3z^{-2} + 4z^{-3})$$
$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

↓

$$y[n] = \delta[n-1] + \delta[n-2] + 2\delta[n-3] + 2\delta[n-4] - 3\delta[n-5] +$$
$$\delta[n-6] - 4\delta[n-7]$$

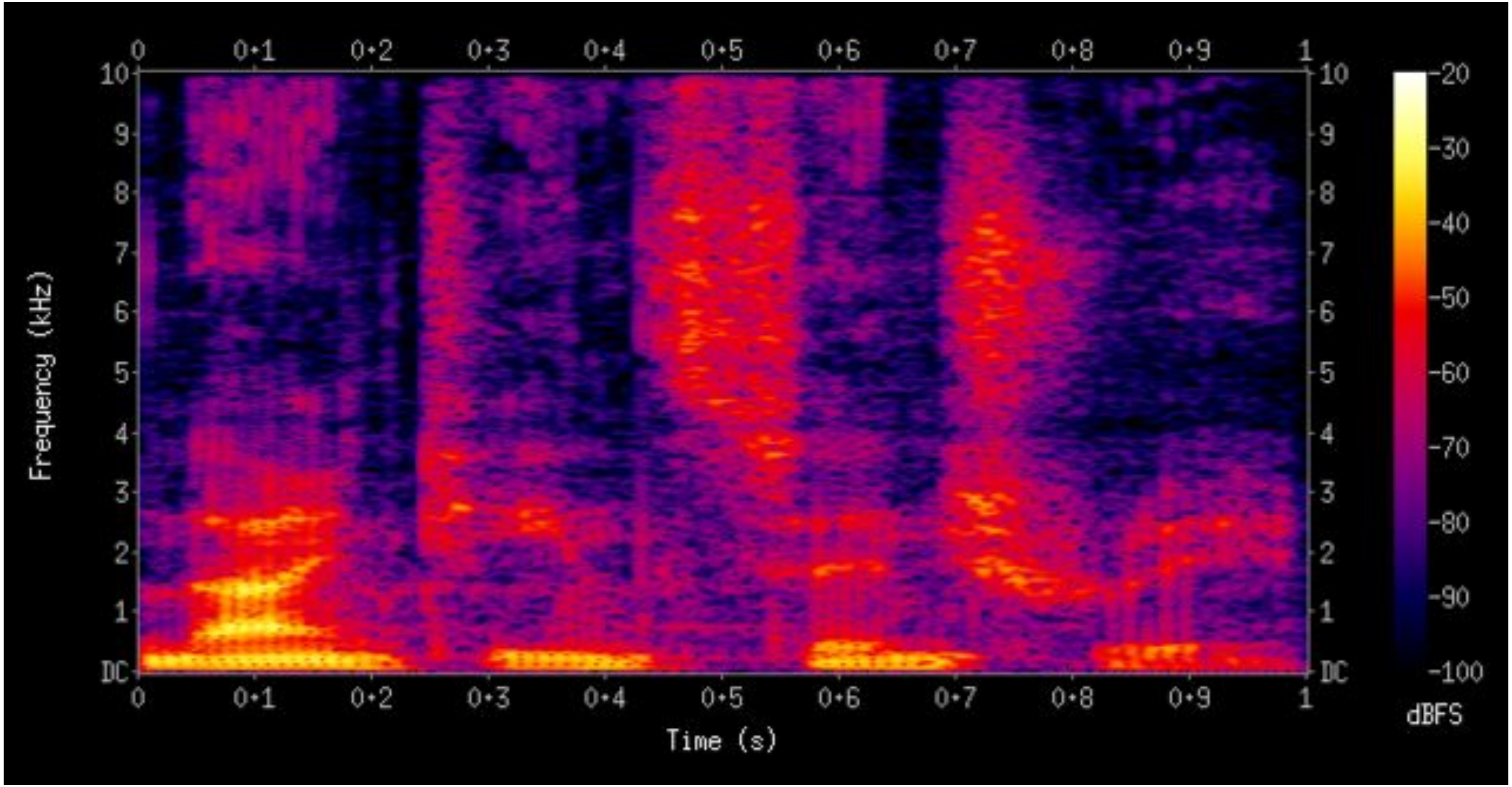


Convolution and Cross-Correlation

TABLE 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time reversal	$f(-t)$	$F(-\omega)$
Time scaling	$f(at)$	$\frac{1}{\|a\|}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\|a\|}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
Modulation (Multiplication)	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
Differentiation in time	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
Differentiation in Frequency	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Symmetry	$f(t)$ real	$F(-\omega) = F^*(\omega)$

Algebraic Properties of Fourier Transform



Frequency (kHz)
Time (s)
dBFS
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1
DC
10
-20
-30
-40
-50
-60
-70
-80
-90
-100

Exercises from Last Time

- A vuvuzela produces 116 dB at 1m. How loud is it a soccer field away in dB? How loud would it be if there were 20,000 people in a stadium playing vuvuzelas at that distance?

Soccer field is ~100m. By $1/r^2$, 116dB - $10*\log_{10}((1m/100m)^2)$ = 116dB - 40dB = 76dB
Now, if there are 20,000 of them: 76dB + $10*\log_{10}(20{,}000)$ = 76dB + 43dB = 119dB
Hearing loss occurs at 120dB!!

- In the trenches of WW1, on September 28, 1915, German artillery in Belgium could be heard more than sixty miles away, however not between thirty and sixty miles away. Why not?

A thermal gradient caused distant sound to refract over a dead zone.

- A voiced consonants uses the vocal cords. You can tell if a sound is voiced by touching your throat when you make the sound. /z/ (as in "zinc") is the voiced version of the /s/ (as in "sink") alveolar fricative. What is the voiced version of the palato-alveolar fricative /ʃ/ (as in "ship")?

The voiced palato-alveolar fricative is /ʒ/ as in "pleasure" and "vision".

Exercises for Next Time

- If someone is whistling at a frequency of 16kHz, and you're recording them with a sampling rate of 20kHz, what frequencies could the 16kHz signal appear as?
- What sampling rate would you need to use to make sure the 16kHz frequencies are captured exactly?
- Construct the following for the signal $y[n] = x[n] * h[n] = 3x[n] + 5x[n-1]$, where $x[n]$ is an input signal and $h[n]$ is the transfer function of our filter:
    - $h[n] =$
    - $H(z) =$